

Two-Loop Diagrams in Causal Perturbation Theory

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Scalar two-loop diagrams are calculated analytically in massive cases needed for the computation of boson and fermion propagators in QED and QCD by the causal method of Epstein and Glaser. It is demonstrated that this method, which is the basis for the so-called dispersive methods, provides many advantages for higher loop calculations in comparison to the usual Feynman integrals. It is possible to express the propagators as double integrals in a straightforward manner. In the case of vacuum polarization both integrations can be carried out in terms of polylogarithms, whereas the last integral in the fermion propagator cannot be expressed by known special functions. The infrared problems in connection with the adiabatic limit are carefully discussed. © 1997 Academic Press

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1. THE CAUSAL APPROACH TO QUANTUM FIELD THEORY

1.1. Introduction

In the traditional approach to quantum field theory, one usually starts from classical fields and a Lagrangean which describes the interaction. These objects get quantized and S-matrix elements or Greens functions are constructed with the help of the famous Feynman rules. However, the Feynman rules are not mathematically well-defined since they contain the product of operator-valued distributions with discontinuous step functions. This leads to the well-known ultraviolet-divergencies

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in perturbation theory. The occurrence of these divergencies has led to the invention of several ingenious formal procedures like Pauli-Villars regularization [1] or dimensional regularization [2] which explicitly isolate the finite expressions from the divergent ones. It is then possible to eliminate the divergencies by a redefinition of physical parameters of the theory. But still the situation is not entirely satisfactory. So Feynman in his Nobel lecture remarked: "I think that the renormalization theory is simply a way to sweep the difficulties of the divergencies of electrodynamics under the rug."

However, it is possible to avoid ultraviolet-divergencies in perturbation theory from the beginning by following Epstein and Glaser's causal approach to quantum field theory that explicitly uses the causal structure of the theory as a powerful tool [3]. In the causal theory the perturbative S-matrix is viewed as an operator-valued distribution [4, 5] of the form

$$S(g) = \mathbf{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 \cdots dx_n T_n(x_1, \dots, x_n) g(x_1) \cdots g(x_n), \quad (1.1)$$

where $g \in \mathcal{S}(\mathbf{R}^4)$, the Schwartz space of functions of rapid decrease. The perturbation series (1.1) is purely formal and we cannot make any statement about the convergence of this series. The test function g plays the role of 'adiabatic switching' and provides a cutoff in the long-range part of the interaction, without destroying any symmetry. It can be considered as a natural infrared regulator. The adiabatic limit $g \rightarrow 1$ must be performed at the end of calculations in the right quantities where this limit exists [6]. The existence becomes a problem if the theory contains massless fields. Only well-defined quantities, namely free asymptotic fields acting on the Fock space, are utilized to construct $S(g)$.

Let us choose two test functions which have disjoint supports in time, i.e.,

$$\begin{aligned} \text{supp } g_1 &\subset \{x \in \mathbf{M}_m \mid x^0 \in (-\infty, r)\}, \\ \text{supp } g_2 &\subset \{x \in \mathbf{M}_m \mid x^0 \in (r, \infty)\}. \end{aligned} \quad (1.2)$$

$\mathbf{M}_m \equiv (\mathbf{R}^m, \eta)$ denotes the m -dimensional Minkowski space with the metric tensor

$$\eta = \text{diag}(1, -1, \dots, -1). \quad (1.3)$$

We express the special relation between g_1 and g_2 by writing $\text{supp } g_1 < \text{supp } g_2$. Then Epstein and Glaser require

$$S(g_1 + g_2) = S(g_2) S(g_1). \quad (1.4)$$

This causal factorization condition can be transformed into a recursion condition for the T_n (2.16, 2.17). If the support property of the lowest order commutator

$$\text{supp}[T_1(x), T_1(y)] \subseteq \{x, y \in \mathbf{M}_m \mid (x-y)^2 \geq 0\} \quad (1.5)$$

is satisfied, then all $T_n(x_1, \dots, x_n)$ can be constructed inductively order by order. However, the induction procedure doesn't fix the T_n completely; they are only determined up to distributions with support Δ_n , where Δ_n is the complete diagonal ($x_1 = \dots = x_n$) in $\mathbf{M}_m^{\otimes n}$. But this ambiguity can be further restricted by additional physical conditions.

In this paper, we will show that the causal method is also well suited for practical calculations also in higher orders of perturbation theory. Therefore, in Chap. 3 and 4 we will derive analytic expressions for two-loop diagrams in a very compact way.

The considerations above will be reformulated in the next chapter more precisely in a manner which allows a convenient access to practical calculations.

2. THE CAUSAL CONSTRUCTION OF PERTURBATIVE QUANTUM FIELD THEORY

2.1. General Properties of the Causal Method

In contrast to the usual approaches, the causal method yields the perturbative scattering matrix $S(g)$ directly in the Fock space of the well-defined asymptotic free fields (which may also contain an unphysical sector) and thus avoids the problematic notion of a quantized interacting field. Some basic physical assumptions are needed in order to construct the S-matrix.

Causality. If the support of the test function $g_1 \in \mathcal{S}(\mathbf{R}^4)$ in Minkowski space is earlier than $\text{supp } g_2$ in some Lorentz frame ($\text{supp } g_1 < \text{supp } g_2$), then the S-matrix fulfills the following functional equation:

$$S(g_1 + g_2) = S(g_2) S(g_1). \quad (2.1)$$

Poincaré invariance. Given the usual (not necessarily unitary) representation $U(a, A)$ of the Poincaré group P_+^\uparrow in the Fock space, then the condition of *Poincaré invariance* of the S-matrix can be formulated as follows:

$$U(a, \mathbf{1}) S(g) U(a, \mathbf{1})^{-1} = S(g_a) \quad \forall a \in \mathbf{R}^4, \quad (2.2)$$

where

$$g_a(x) = g(x - a) \quad (\text{Translational invariance}), \quad (2.3)$$

and

$$U(0, A) S(g) U(0, A)^{-1} = S(g_A) \quad \forall A \in L_+^\uparrow \cong SO^+(1, 3), \quad (2.4)$$

where

$$g_A(x) = g(A^{-1}x) \quad (\text{Lorentz invariance}). \quad (2.5)$$

Interaction. The *specific coupling* $T_1(x)$ of the studied theory is given in (1.1) in terms of free fields.

EXAMPLES. U(1)-gauge theory (QED) [7]:

$$T_1(x) = ie : \bar{\Psi}(x) \gamma^\mu \Psi(x) : A_\mu(x). \quad (2.6)$$

SU(N)-gauge theory without matter fields [8]:

$$T_1 = ig f_{abc} : \frac{1}{2} A_\mu^a A_\nu^b F_c^{\nu\mu} - A_\mu^a u_b \partial^\mu \tilde{u}_c :. \quad (2.7)$$

Scalar QED [9]:

$$T_1 = e : \varphi^\dagger \overleftrightarrow{\partial}_\mu \varphi : A^\mu$$

Gravity [10]:

$$\begin{aligned} T_1 = & \frac{iG}{2} : h^{\rho\sigma} (h^{\alpha\beta}{}_{,\rho} h_{\alpha\beta}{}_{,\sigma} - \frac{1}{2} h^\mu{}_{\mu,\rho} h^\nu{}_{\nu,\sigma} + 2h_{\sigma\beta,\alpha} h_\rho{}^{\alpha,\beta} + h^\mu{}_{\mu,\alpha} h_{\rho\sigma}{}^{,\alpha} - 2h_{\alpha\rho,\beta} h_\sigma{}^{\alpha,\beta}) : \\ & + \frac{iG}{2} : \tilde{u}_\mu h^{\mu\nu}{}_{,\nu} u^\rho{}_{,\rho} + 2\tilde{u}_{\mu,\rho} h^{\nu\rho} u^\mu{}_{,\nu} :. \end{aligned} \quad (2.9)$$

Totally scalar QED:

$$T_1 = ie : \varphi^\dagger \varphi : A - h.c. \quad (2.10)$$

The free fields in here are defined in the natural manner by the aid of their wave equations and commutation relations [Appendix A]. It is worth mentioning that the 4-boson couplings appearing in the Lagrangean for QCD, Gravity and scalar QED are absent in T_1 ; in fact they are automatically generated in second order by an asymptotic version of gauge invariance in the causal theory [7, 8, 10]. A further important remark is concerning unitarity: Since the construction of a Lorentz covariant quantum field theory often leads to the introduction of Fock spaces with “unphysical sectors” (e.g. ghost-sectors), unitarity cannot be formulated within the theory in terms of a naive requirement like $S(g)^{-1} = S(g)^\dagger$.

The inverse S -matrix is expressed analogously to (1.1) by a formal power series:

$$\begin{aligned} S(g) &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 \cdots dx_n \tilde{T}_n(x_1, \dots, x_n) g(x_1) \cdots g(x_n) \\ &= (\mathbf{1} + T)^{-1} = \mathbf{1} + \sum_{r=1}^{\infty} (-T)^r. \end{aligned} \quad (2.11)$$

The $\tilde{T}_n(x_1, \dots, x_n)$ are symmetric in x_1, \dots, x_n , therefore we use the set-theoretical notation $X = \{x_1, \dots, x_n\}$. The distributions \tilde{T}_n can be computed by formal inversion of (1.1):

$$\tilde{T}_n(X) = \sum_{r=1}^{\infty} (-1)^r \sum_{P_r} T_{n_1}(X_1) \cdots T_{n_r}(X_r), \quad (2.12)$$

where the second sum runs over all partitions P_r of X into r disjoint subsets

$$X = X_1 \cup \cdots \cup X_r, \quad X_j \neq \emptyset, \quad |X_j| = n_j. \quad (2.13)$$

Furthermore one can deduce two important relations for the n -point distributions by carrying out the formal multiplication of the expansion of the S-matrix with its inverse,

$$\sum_{P_2^o} T_{n_1}(X) \tilde{T}_{n-n_1}(Z \setminus X) = 0, \quad (2.14)$$

for $|Z| = n \geq 1$, $|X| = n_1$, and

$$\sum_{P_2^o} \tilde{T}_{n-n_2}(Y) T_{n-n_2}(Z \setminus Y) = 0, \quad (2.15)$$

for $|Z| = n \geq 1$, $|Y| = n_2$.

The idea of constructing the S-matrix with the help of adiabatic switching goes back to N. N. Bogoliubov [11] and collaborators. They translated the above conditions imposed on S(g) into conditions on the n -point distributions $T_n(x_1, \dots, x_n)$ and $\tilde{T}_n(x_1, \dots, x_n)$:

Causality:

$$T_n(x_1, \dots, x_n) = T_m(x_1, \dots, x_m) T_{n-m}(x_{m+1}, \dots, x_n),$$

$$\text{if } \{x_{m+1}, \dots, x_n\} < \{x_1, \dots, x_m\}, \quad (2.16)$$

$$\tilde{T}_n(x_1, \dots, x_n) = \tilde{T}_m(x_1, \dots, x_m) \tilde{T}_{n-m}(x_{m+1}, \dots, x_n),$$

$$\text{if } \{x_1, \dots, x_m\} < \{x_{m+1}, \dots, x_n\}. \quad (2.17)$$

Poincaré invariance:

$$U(a, A) T_n(x_1, \dots, x_n) U(a, A)^{-1} = T_n(Ax_1 + a, \dots, Ax_n + a) \quad \forall (a, A) \in P_+^\uparrow. \quad (2.18)$$

If case of a unitary S-matrix a further property of the T_n , \tilde{T}_n can be added:

$$\tilde{T}_n(x_1, \dots, x_n) = T_n^\dagger(x_1, \dots, x_n). \quad (2.19)$$

The crucial point in the causal formulation of perturbation theory is that the usual formal definition of the T_n via simple time-ordering

$$T_n(x_1, \dots, x_n) = T\{T_1(x_1) \cdot \dots \cdot T_1(x_n)\} \\ \equiv \sum_{\Pi} \Theta(x_{\Pi_1}^o - x_{\Pi_2}^o) \cdot \dots \cdot \Theta(x_{\Pi_{n-1}}^o - x_{\Pi_n}^o) T_1(x_{\Pi_1}) \cdot \dots \cdot T_1(x_{\Pi_n}), \quad (2.20)$$

where the sum runs over all $n!$ permutations, contains ultraviolet divergencies, therefore there must be an error in the derivation. Epstein and Glaser proceed more carefully and introduce the following n -point distributions:

$$A'_n(x_1, \dots, x_n) = \sum_{P_2} \tilde{T}_{n_1}(X) T_{n-n_1}(Y, x_n), \quad (2.21)$$

$$R'_n(x_1, \dots, x_n) = \sum_{P_2} T_{n-n_1}(Y, x_n) \tilde{T}_{n_1}(X); \quad (2.22)$$

here the sums run over all partitions

$$P_2: \{x_1, \dots, x_{n-1}\} = X \cup Y, \quad X \neq \emptyset$$

into disjoint subsets with $|X| = n_1$, $|Y| \leq n - 2$. Assuming by induction that T_1, \dots, T_{n-1} are known, then A'_n and R'_n can be calculated. One also introduces

$$D_n(x_1, \dots, x_n) = R'_n - A'_n. \quad (2.23)$$

If the sums are extended over all partitions P_2^0 , including the empty set $X = \emptyset$, we obtain the distributions

$$A_n(x_1, \dots, x_n) = \sum_{P_2^0} \tilde{T}_{n_1}(X) T_{n-n_1}(Y, x_n) \\ = A'_n + T_n(x_1, \dots, x_n), \quad (2.24)$$

$$R_n(x_1, \dots, x_n) = \sum_{P_2^0} T_{n-n_1}(Y, x_n) \tilde{T}_{n_1}(X) \\ = R'_n + T_n(x_1, \dots, x_n). \quad (2.25)$$

These two distributions are not known by the induction assumption because they contain the unknown T_n . Only the difference

$$D_n = R'_n - A'_n = R_n - A_n \quad (2.26)$$

is known. We stress the fact that all products of distributions in here are well-defined because the arguments are disjoint sets of points so that the products are tensor products of distributions.

One can determine R_n or A_n separately by investigating the support properties of the various distributions, this is the point where the causal structure becomes important. It turns out that R_n is a retarded and A_n an advanced distribution [7]

$$\text{supp } R_n \subseteq \bar{F}_{n-1}^+(x_n), \quad \text{supp } A_n \subseteq \bar{F}_{n-1}^-(x_n), \quad (2.27)$$

with

$$\begin{aligned} \bar{F}_{n-1}^\pm(x) &\equiv \{(x_1, \dots, x_{n-1}) \mid x_j \in \bar{V}^\pm(x), \forall j = 1, \dots, n-1\}, \\ \bar{V}^\pm(x) &= \{y \mid (y-x)^2 \geq 0, \pm(y^0 - x^0) \geq 0\}. \end{aligned} \quad (2.28)$$

Hence, by splitting of the causal distribution (2.26) one gets R_n (and A_n), and T_n then follows from (2.24) (or (2.25)). The T_n 's so obtained are well-defined time-ordered products. Local terms with support $(x_1 = \dots = x_n)$, originating from a certain ambiguity in the splitting procedure (see next chapter), might spoil the symmetry of the T_n 's in x_1, \dots, x_n , but this minor problem can be removed by subsequent symmetrization.

To carry out the splitting process, we write (2.26) in normally ordered form and split the numerical distributions $d_n^k(x)$, where $x = (x_1 - x_n, \dots, x_{n-1} - x_n)$

$$D_n(x_1, \dots, x_n) = \sum_k d_n^k(x_1 - x_n, \dots, x_{n-1} - x_n) : \mathcal{O}_k(x_1, \dots, x_n) :. \quad (2.29)$$

\mathcal{O}_k : is a normally ordered product of external field operators (Wick monomial). It is a consequence of translation invariance that $d_n^k(x)$ depends only on relative coordinates.

The only nontrivial step in the construction of well-defined time-ordered products is the splitting of a numerical distribution d with support in $\bar{F}^+ \cup \bar{F}^-$ into a distribution r with support in \bar{F}^+ and a distribution a with support in \bar{F}^- . In causal perturbation theory the usual formal time-ordered products with subsequent renormalization are replaced by this conceptually simple and mathematically well-defined procedure. In fact the problem of distribution splitting was already solved in a general framework by the mathematician Malgrange in 1960 [12]. Epstein and Glaser used his general result for the special case of relativistic quantum field theory [3]. In the next chapter we present a new formulation of the splitting problem which provides a simple solution in the case of quantum field theory with massive fields.

2.2. The Theory of Distribution Splitting

Consider a tempered numerical distribution $d \in \mathcal{S}'(\mathbf{R}^n \cong \mathbf{M}_4^{\otimes m})$, $n = 4m$, with support in the cone $\bar{F}_m^+(0) \cup \bar{F}_m^-(0)$. The splitting problem can be formulated as follows: Is it possible to find a pair (r, a) of tempered distributions on $\mathbf{M}_4^{\otimes m}$, with

supports in $\bar{I}_m^+(0)$ and $\bar{I}_m^-(0)$, respectively, such that $r - a = d$? The answer is affirmative; indeed there exists a general theory of dissecting distributions into two parts with prescribed supports, provided these supports are “regularly separated” [12]. But the special problem in quantum field theory is much simpler, especially in the case where exclusively massive fields are interacting.

We first note that if there are two solutions (r_1, a_1) and (r_2, a_2) , then the difference $r_1 - r_2 = a_1 - a_2$ has support $\{0\}$ and must therefore be of the form [5]

$$\sum_{|\alpha|=0}^M c_\alpha D^\alpha \delta(x), \quad D^\alpha = \frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial y_1^{\alpha_1} \dots \partial y_n^{\alpha_n}},$$

where

$$x_j^\mu = y_{4(j-1) + \mu + 1}, \quad \mu = 0, \dots, 3, \quad \text{or} \quad x = (x_1, \dots, x_m) = (y_1, \dots, y_n). \quad (2.30)$$

The behaviour of the distribution at $x=0$ is obviously decisive for the splitting problem. We therefore classify the singularities of distributions in this region. This is best done with the singular order of causal distributions which is a rigorous definition of the power-counting degree [7].

DEFINITION. The distribution $d(x) \in \mathcal{S}'(\mathbf{R}^n)$ has *quasi-asymptotics* $d_0(x)$ at $x=0$ with regard to a positive continuous function $\rho(\delta)$, $\delta > 0$, if the limit

$$\lim_{\delta \rightarrow 0} \rho(\delta) \delta^n d(\delta x) = d_0(x) \neq 0$$

exists in $\mathcal{S}'(\mathbf{R}^n)$.

The equivalent definition in momentum space reads as follows:

DEFINITION. The distribution $\hat{d}(p) \in \mathcal{S}'(\mathbf{R}^n)$ has quasi-asymptotics $\hat{d}_0(p)$ at $p = \infty$ with regard to a positive continuous function $\rho(\delta)$, $\delta > 0$, if the limit

$$\lim_{\delta \rightarrow 0} \rho(\delta) \left\langle \hat{d} \left(\frac{p}{\delta} \right), \check{\phi}(p) \right\rangle = \langle \hat{d}_0, \check{\phi} \rangle \quad (2.32)$$

exists for all $\check{\phi} \in \mathcal{S}(\mathbf{R}^n)$. The Fourier transform of a test function $\varphi(x)$ is defined by

$$\hat{\varphi}(p) = (2\pi)^{-n/2} \int d^4x_1 \dots d^4x_m \varphi(x_1, \dots, x_m) e^{ip_1x_1 + \dots + ip_mx_m}. \quad (2.33)$$

By scaling transformation one derives

$$\lim_{\delta \rightarrow 0} \frac{\rho(a\delta)}{\rho(\delta)} = a^\omega \equiv \rho_\omega(\delta) \quad (2.34)$$

with some real ω . Thus we call $\rho(\delta)$ the power-counting function.

DEFINITION. The distribution $d(x) \in \mathcal{S}'(\mathbf{R}^n)$ is called *singular of order ω* , if it has a quasi-asymptotics $d_0(x)$ at $x=0$, or its Fourier transform has quasi-asymptotics $\hat{d}_0(p)$ at $p = \infty$, respectively, with power-counting function $\rho(\delta)$ satisfying

$$\lim_{\delta \rightarrow 0} \frac{\rho(a\delta)}{\rho(\delta)} = a^\omega \quad \forall a > 0. \quad (2.35)$$

Eq. (2.34) implies

$$\begin{aligned} a^n \langle \hat{d}_0(p), \check{\varphi}(ap) \rangle &= \left\langle \hat{d}_0\left(\frac{p}{a}\right), \check{\varphi}(p) \right\rangle \\ &= a^{-\omega} \langle \hat{d}_0(p), \check{\varphi}(p) \rangle \\ &= \left\langle d_0(x), \varphi\left(\frac{x}{a}\right) \right\rangle \\ &= a^n \langle d_0(ax), \varphi(x) \rangle \\ &= a^{-\omega} \langle d_0(x), \varphi(x) \rangle, \end{aligned} \quad (2.36)$$

i.e., \hat{d}_0 is homogeneous of degree ω :

$$\hat{d}_0\left(\frac{p}{a}\right) = a^{-\omega} \hat{d}_0(p), \quad (2.37)$$

$$d_0(ax) = a^{-(n+\omega)} d_0(x). \quad (2.38)$$

This implies that d_0 has quasi-asymptotics $\rho(\delta) = \delta^\omega$ and the singular order ω , too. In particular, we have the following estimates for $\rho(\delta)$ [7]: If $\varepsilon > 0$ is an arbitrarily small number, then there exist constants C , C' and δ_0 such that

$$C\delta^{\omega+\varepsilon} \geq \rho(\delta) \geq C'\delta^{\omega-\varepsilon}, \quad \delta < \delta_0. \quad (2.39)$$

We illustrate the notions introduced so far by giving some examples:

- (1) $d = 1 \in \mathcal{S}'(\mathbf{R}^n)$: We get $\rho(\delta) = \delta^{-n}$, $\omega = -n$ and $d_0 = 1$.
- (2) $d(y_1, \dots, y_n) = D^\alpha \delta(y_1, \dots, y_n)$ where

$$\begin{aligned} D^\alpha &= \frac{\partial^{\alpha_1 + \dots + \alpha_n}}{\partial y_1^{\alpha_1} \dots \partial y_n^{\alpha_n}}, & |\alpha| &= \alpha_1 + \dots + \alpha_n, \\ \rho(\delta) &= \delta^{|\alpha|} & \text{and} & \quad \omega = |\alpha|. \end{aligned} \quad (2.40)$$

$$(3) \quad d(x) = \Theta(1 - x^2) \log |x| \in \mathcal{S}'(\mathbf{R}):$$

$$\rho(\delta) = \frac{c}{\delta \log \delta}, \quad c \neq 0,$$

$$\begin{aligned} \lim_{\delta \rightarrow 0} \frac{c}{\delta \log \delta} \delta(\log \delta + \log |x|) \Theta(1 - \delta^2 x^2) \\ = \lim_{\delta \rightarrow 0} c \left(1 + \frac{\log |x|}{\log \delta} \right) = c = d_0(x), \end{aligned} \quad (2.41)$$

$$\rho_0(a) = \lim_{\delta \rightarrow 0} \frac{c}{a\delta(\log \delta + \log a)} \frac{\delta \log \delta}{c} = a^{-1} \Rightarrow \omega = -1.$$

Originally, Epstein and Glaser introduced a definition that differs from the one given above. But their definition is suffering from the fact that the corresponding definitions in x - and p -space are not completely equivalent.

Now we are capable to construct explicit splitting solutions (for complete proofs see [3, 7]). First we specify the splitting problem by requiring that the splitting procedure preserves the singular order of the distributions in the following sense:

$$\omega(r) \leq \omega(d) \wedge \omega(a) \leq \omega(d). \quad (2.42)$$

Then we have to distinguish two cases:

Case 1: $\omega < 0$. In this case, the power counting function tends to infinity

$$\rho(\delta) \rightarrow \infty \quad \text{for } \delta \rightarrow 0. \quad (2.43)$$

This implies

$$\left\langle d(x), \varphi \left(\frac{x}{\delta} \right) \right\rangle \rightarrow \frac{\langle d_0, \varphi \rangle}{\rho(\delta)} \rightarrow 0. \quad (2.44)$$

Now we choose an arbitrary but fixed vector $v = (v_1, \dots, v_m) \in \Gamma_m^+(0)$, which means that all four-vectors are inside the forward light-cone V^+ . The $(4m - 1)$ -dimensional hyperplane, defined by

$$vx = \sum_{j=1}^m v_j x_j = 0, \quad (2.45)$$

splits the causal support; all products $v_j x_j$ are either > 0 for $x \in \Gamma^+$ or < 0 for $x \in \Gamma^-$. In addition, we choose a C^∞ -function χ_0 with

$$\chi_0(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ < 1 & \text{for } 0 < t < 1. \\ 1 & \text{for } t \geq 1 \end{cases} \quad (2.46)$$

It can be shown [7] that the limit

$$\lim_{\delta \rightarrow 0} \chi_0 \left(\frac{vx}{\delta} \right) d(x) \equiv \Theta(vx) d(x) = r(x) \quad (2.47)$$

exists, defining unambiguously the multiplication of $d(x)$ by a Θ step-function. One arrives at $r(x)$ with $\text{supp } r(x) \subseteq \bar{\Gamma}_m^+(0)$. The advanced part is then given by

$$a(x) = r(x) - d(x), \quad \text{supp } a(x) \subseteq \bar{\Gamma}_m^-(0). \quad (2.48)$$

It is not difficult to verify that the singular order does not increase in the splitting process. Adding distributions with point-support in $\{0\}$ to r or a would violate the uniqueness and scaling behaviour of r and a (2.42).

Since all explicit calculations in quantum field theory are best done in momentum space, we must study the splitting procedure in p -space. For this reason, we remember that the product in x -space of a tempered distribution and a test function in Schwartz space goes over into a convolution in p -space:

$$\widehat{fg}(p) = (2\pi)^{-n/2} \langle f(x), g(x) e^{ipx} \rangle, \quad f \in \mathcal{S}'(\mathbf{R}^n), \quad g \in \mathcal{S}(\mathbf{R}^n). \quad (2.49)$$

The naive application of this statement to the product $\Theta(vx) d(x)$ leads to the correct result, if the theory contains no massless fields (this can be proven rigorously [7]). For the purpose of this, we consider time-like $p \in \Gamma^+$. We choose a special coordinate system such that $p = (p_1^0, \mathbf{0}, 0, \dots)$. Note that this coordinate system is obtained from the original one by an orthogonal transformation $\in SO(4m)$ and not by a Lorentz transformation. Furthermore we take v parallel to p , i.e. $v = (1, \mathbf{0}, 0, \dots)$. Then v varies with p , but this is admissible because $\Theta(vx) d(x)$ is actually independent of v . Now we have $\Theta(vx) = \Theta(x_1^0)$ and the Fourier transform of $\Theta(x_1^0)$ is given by

$$\hat{\Theta}(p) = (2\pi)^{2m-1} \delta(\mathbf{p}_1, p_2, \dots, p_m) \frac{i}{p_1^0 + i0}. \quad (2.50)$$

This leads to the dispersion relation

$$\hat{r}(p_1^0) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dk_0 \frac{\hat{d}(k_0)}{p_1^0 - k_0 + i0}. \quad (2.51)$$

To write down the result for arbitrary $p \in \Gamma^+ \cup \Gamma^-$, we use the variable of integration $t = k_0/p_1^0$ and arrive at

$$\hat{r}(p) = \pm \frac{i}{2\pi} \int_{-\infty}^{\infty} dt \frac{\hat{d}(tp)}{1 - t \pm i0}, \quad p \in \Gamma^\pm. \quad (2.52)$$

Case 2: $\omega \leq 0$. Now the power-counting function satisfies

$$\frac{\rho(\delta)}{\delta^{\omega+1}} \rightarrow \infty \quad \text{for } \delta \rightarrow 0. \quad (2.53)$$

The careless multiplication of $d(x)$ by $\Theta(vx)$ in p -space would lead to an *ultraviolet divergent expression*. Nevertheless, it is possible to derive a subtracted dispersion relation for $r_0(p)$:

$$\hat{r}_0(p) = \pm \frac{i}{2\pi} \int_{-\infty}^{\infty} dt \frac{\hat{d}(tp)}{(t \mp i0)^{\omega+1} (1-t \pm i0)}, \quad p \in \Gamma^\pm. \quad (2.54)$$

If ω is not integer (which seems not to occur in standard theories like QED), we use the largest integer $[\omega] < \omega$ instead of ω . It is easy to verify that this dispersion integral is convergent for $|t| \rightarrow \infty$. In fact, we have

$$\hat{d}(tp) \rightarrow \frac{\hat{d}_0(p)}{\rho(1/t)}, \quad |t| \rightarrow \infty, \quad (2.55)$$

and since the power counting function ρ is bounded by

$$\frac{1}{\rho(1/t)} < \frac{|t|^{\omega+\varepsilon}}{C(\varepsilon)}, \quad \varepsilon < 1, \quad (2.56)$$

the integral is absolutely convergent after smearing out with a test function $\check{\varphi}(p)$. We call (2.54) the central splitting solution, because it is normalized at the origin in p -space by

$$D_p^\alpha \hat{r}_0(0) = 0, \quad |\alpha| \leq \omega. \quad (2.57)$$

A more general but not very handy formula, which is valid for arbitrary p , is

$$\hat{r}_q(p) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{t+i0} \left[\hat{d}(p-tv) - \sum_{|\alpha|=0}^{\omega} \frac{p^\alpha}{\alpha!} (D_q^\alpha \hat{d})(q-tv) \right], \quad (2.58)$$

where $v \in \Gamma^+$ is fixed, and it fulfils $D^z \hat{r}_q(q) = 0$. The central splitting solution (2.54) has two important properties: First, it does not introduce a new mass scale into the theory. If we choose a splitting solution with normalization point $q \neq 0$, then $|q|^2 = M^2$ defines such a scale. Second, most symmetry properties of \hat{d} are preserved under central splitting, because the origin $q = 0$ is a very symmetric point. There is also the possibility to determine $\hat{r}(p)$ by analytic continuation. This method is based on the following well-known theorem [13].

The retarded (advanced) distribution $\hat{r}(p)(\hat{a}(p))$ is the boundary value of an analytic function, regular in $\mathbf{R}^n + i\Gamma^+(\mathbf{R}^n - i\Gamma^+)$.

Due to the fact that the difference between \hat{r}_0 and \hat{r}_q has causal support and taking into account example 2), we mention that

$$\hat{r}_q(p) - \hat{r}_0(p) = \sum_{|\alpha|=0}^{\omega} c_{\alpha}^q p^{\alpha}. \quad (2.59)$$

The freedom in the normalization of retarded distributions with $\omega \geq 0$ by polynomials with coefficients c_{α}^q must be restricted by further physical considerations. In every order n of perturbation theory, there is then a finite number of free parameters. Considering the inductive procedure in n , three possibilities arise: (i) the number of free parameters increases with n without bound; the theory is then called non-normalizable. (ii) the number of free parameters appearing in all orders is finite; the theory is normalizable. (iii) There are only finitely many graphs with $\omega \geq 0$, then the theory is called super-normalizable. The expression “renormalizable”, widely used in literature, is of use in causal perturbation theory in the restricted sense of a finite renormalization which might be necessary to restore certain symmetries of the theory.

To determine in which of the three cases we are, we need more information about the singular behaviour of the numerical distributions appearing in higher orders. An analysis of the inductive construction of the theory shows that one can give an upper bound for the singular order of the distributions in a graph, depending on the external field operators and the order n of the graph. As an example, we find for QED

$$\omega \leq 4 - b - \frac{3}{2}f, \quad (2.60)$$

where b is the number of external photons and f the number of external electrons and positrons. For totally scalar QED, the model theory used in the following chapters, the analogous formula to (2.60) is

$$\omega \leq 4 - n - l. \quad (2.61)$$

Here, l is simply the number of external lines.

3. THE CALCULATION OF TWO-LOOP DIAGRAMS IN CAUSAL PERTURBATION THEORY

3.1. Introduction

A beginner who has taken his first elementary course in the causal method might be a bit sceptical in view of the seemingly large bookkeeping that has to be done in the causal theory in comparison with Feynman rules in order to calculate a Feynman diagram in higher orders. But still, the Feynman rules are not optimal for

loop diagrams. The main reason for this fact is the following. Suppose that time ordering can simply be done by multiplication with step functions $\Theta(x_k^0 - x_j^0)$, as in the scalar theory we are going to consider. Then, in calculating a time-ordered product according to the rules, every pairing of field operators gets such a Θ -function which leads to the Feynman propagators $D_F(x_k - x_j)$. However, the time ordering of the vertices is already specified by less Θ -functions. Without the temporal Θ -function the pairing function is D^+ instead of D_F . Consequently, some propagators D_F can actually be simplified into D^\pm which is simpler to integrate because it contains a δ -distribution in p-space. For example, in the causal theory the third order scalar vertex function comes out to be

$$A(x_1, x_2, x_3) = D_F(x_1 - x_3) D^+(x_3 - x_2) D_F(x_1 - x_2) - D_F D^{av} D^- + D^+ D^{av} D^{ret}. \tag{3.1}$$

If one substitutes $D^+ = D_F - D^{av}$ in the first term, one arrives at the usual Feynman form

$$A = D_F D_F D_F - D_F D^{av} (D_F + D^-) + D^+ D^{av} D^{ret}, \tag{3.2}$$

because the last two terms are equal to $D^{av} D^{av} D^{ret}$ which vanishes by the support properties of the advanced and retarded distributions. But (3.1) is simpler to calculate than (3.2) because every member contains one D^\pm . This advantage is strongly increasing in higher loop diagrams.

It is one of the purposes of this paper to illustrate these features for the so-called two-loop master diagram of Fig.1. Interestingly enough, this diagram was already computed with the causal method by various authors without knowing it. The first were Källén and Sabry [14], their work was extended by Broadhurst [15] to other mass cases. The most extensive higher order calculations using the “dispersive method” were carried out over many years by the italian group Mignaco, Remiddi [16], Barbieri [17] and others (see [18] and references given therein). All these authors base their calculations on analytic properties of Feynman integrals which are often referred to as Cutcosky rules. The lack of understanding of these analytic properties has created the problem of anomalous thresholds. Fortunately, the latter do not appear in diagrams with two and three external legs, but they do appear in four- and more legs-diagrams [19].

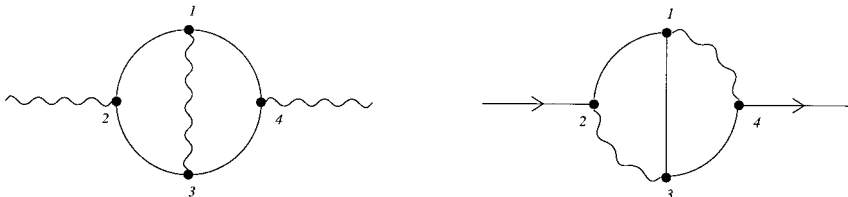


FIG. 1. The master diagram.

As a first application of causal perturbation theory, we will now briefly describe the calculation of the scalar vertex function with arbitrary masses which is needed at various places in the later two-loop calculations. Then we turn to the master diagram for vacuum polarization. We show the calculation of the causal distribution in some detail to localize the infrared divergences which appear in the case of vanishing photon mass. The most difficult integration is the dispersion integral for the splitting. Finally we consider the more complicated case of the fermion propagator. By ingenious tricks, Broadhurst [15] succeeded in expressing this propagator by a one-dimensional integral over the complete elliptic integral of the first kind $K(k)$ together with elementary functions which can easily be computed numerically. In the causal theory one needs not be a genius. We get the result by a slight change of the standard procedure. As far as the spin structure is concerned, it is known that the general two-legs diagram can be expressed in terms of the scalar two-point functions [20]. Therefore, we restrict ourselves to the scalar case in the present study.

To calculate scalar diagrams with arbitrary masses, we start the inductive process from the following first order

$$T_1(x) = i : \varphi_1^\dagger(x) \varphi_2(x) : A(x) - \text{h.c.} = -\tilde{T}_1(x) = -T_1^\dagger(x). \quad (3.3)$$

Here φ_1, φ_2 are charged scalar fields with masses m_1 and m_2 , respectively, and $A(x)$ is a neutral (self-adjoint) scalar field of mass m_3 . All fields are free fields satisfying the following commutation relations

$$[\varphi_j(x), \varphi_j^\dagger(y)] = -iD_{m_j}(x-y), \quad j=1, 2 \quad (3.4)$$

$$[\varphi_j^{(-)}(x), \varphi_j^{\dagger(+)}(y)] = -iD_{m_j}^+(x-y), \quad (3.5)$$

where D_m is the Jordan-Pauli distribution of mass m and (\pm) refers to the positive and negative frequency parts of the various quantities. $A(x)$ fulfills the same commutation relations without the hermitian adjoint † . The commutator (3.5) gives the contraction in Wick's theorem, its Fourier transform is equal to

$$\hat{D}_m^+(p) = \frac{i}{2\pi} \Theta(p^0) \delta(p^2 - m^2). \quad (3.6)$$

Apparently, the last vertex x_n plays a special role in the above equations. It is the splitting vertex, defining the edge of the causal cone. As already pointed out, by translation invariance the numerical distributions depend only on the relative coordinates

$$y_j = x_j - x_n, \quad j = 1, \dots, n-1. \quad (3.7)$$

The Fourier transform is always understood with respect to these relative coordinates

$$\hat{d}(p) = (2\pi)^{-2n+2} \int d(y) e^{ipy} d^4 y_1 \cdots d^4 y_{n-1}. \quad (3.8)$$

Until now all n th order distributions depend on n or $n-1$ variables, the inner vertices are not integrated out. If the adiabatic limit exists, we can integrate the inner coordinates with $g(x) = 1$. In p -space this means that the inner momenta are put equal to 0. Then many terms in

$$A'_n(x_1, \dots, x_n) = \sum_{P_2} \tilde{T}_{n_1}(X) T_{n-n_1}(Y, x_n), \quad (3.9)$$

$$R'_n(x_1, \dots, x_n) = \sum_{P_2} T_{n-n_1}(Y, x_n) \tilde{T}_{n_1}(X), \quad (3.10)$$

vanish:

LEMMA 1. *In the adiabatic limit only those partitions $X \cup Y$ contribute to A'_n (3.9) where X and $\{Y, x_n\}$ contain external vertices, and similarly for R'_n (3.10).*

Proof. Consider a partition where X contains no external vertex. Performing the contractions between X and $\{Y, x_n\}$ with D^+ -distributions and transforming into p -space (2.17), we get a product of Θ -functions

$$\Theta(p_1'^0) \Theta(p_2'^0) \cdots \Theta(p_j'^0),$$

where all momenta add up to 0:

$$p_1'^0 + p_2'^0 + \cdots + p_j'^0 = 0.$$

Such a product is zero.

There exists one serious problem, however. The central splitting solution (2.52) is only true if all momenta p_j are *inside* the light cone. Therefore, strictly speaking, we cannot put the inner momenta equal to 0. But if the D_m^+ -distribution is massive $m > 0$, the vanishing of the contribution of a wrong partition takes place for small enough inner momenta \tilde{p}_j in V^+ , already. Then we can use (2.52) and take the limit $\tilde{p}_j \rightarrow 0$. For this reason we always calculate with massive fields first. If the limit $m \rightarrow 0$ is required, it must be carefully performed by taking cancellations of infrared divergences between different terms into account.

From Lemma 1 it is clear that in diagrams with two and three external legs, the non-vanishing terms (in the adiabatic limit) correspond to ordinary cuts through the diagram. But in a four-legs diagram a pair of opposite legs can be in X and the other pair of opposite legs in $\{Y, x_n\}$. Then this decomposition is no longer a simple cut and an “anomalous threshold” appears. To our knowledge such a diagram was never computed by the naive “dispersive method”, but in the causal

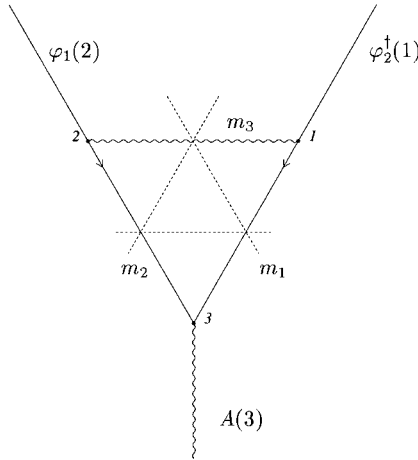


FIG. 2. The vertex diagram.

theory this is no problem. The following second lemma further simplifies the later calculations. It is a consequence of parity- and time-reversal invariance.

LEMMA 2. *In a PT-invariant theory the numerical distributions $d_n^k(x)$ in D_n are essentially real (i.e. up to an overall factor i) in momentum space: $\hat{d}_n^k(p)^* = \hat{d}_n^k(p)$.*

Proof. We know that the causal D -distributions are PT-invariant ([7, p. 281]). This implies for the numerical distributions: $d_n^k(-x) = d_n^k(x)^*$. The complex conjugate comes from the antiunitarity of time-reversal. After Fourier transformation this gives the desired result in momentum space. The overall factor i depends on whether the number of internal lines in the diagram is even or odd.

The formulation of Lemma 2 is of general character. There is a more direct way to understand the lemma in our model theory. The specific coupling has the property $\tilde{T}_1(x) = T_1^\dagger(x)$, and it is not difficult to see that in case of correct normalization of higher orders this leads to a unitary S-matrix: $\tilde{T}_n = T_n^\dagger$. From (2.14, 2.15) we have

$$\begin{aligned} & \sum_{P_2} T^\dagger(Y, x_n) T(X) + \sum_{P_2} T^\dagger(X) T(Y, x_n) \\ &= \sum_{P_2} T(Y, x_n) T^\dagger(X) + \sum_{P_2} T(X) T^\dagger(Y, x_n) = 0, \end{aligned} \quad (3.11)$$

and therefore $D_n = -D_n^\dagger$. Consequently, the d_n^k have to be essentially real.

We now come to the calculation of the third order vertex diagram (Fig. 2). To simplify the notation, we write the arguments in x -space without the dummy variable x . From (3.10) we have

$$R_3' = T_2(1, 3) \tilde{T}_1(2) + T_2(2, 3) \tilde{T}_1(1) + T_1(3) \tilde{T}_2(1, 2). \quad (3.12)$$

The first term herein contains a Compton subgraph

$$R'_{31} = i : \varphi_2^\dagger(1) D_{m_1}^F(1-3) \varphi_2(3) : : A(1) A(3) : (-i) A(2) : \varphi_2^\dagger \varphi_1(2) :, \quad (3.13)$$

where

$$\hat{D}_m^F(p) = \frac{-(2\pi)^{-2}}{p^2 - m^2 + i0} \quad (3.14)$$

is the Feynman propagator. The product (3.13) is computed by Wicks theorem, restricting ourselves to those contractions which generate the vertex diagram:

$$R'_{31} = - : \varphi_2^\dagger(1) D_{m_1}^F(1-3) D_{m_2}^+(3-2) D_{m_3}^+(1-2) \varphi_1(2) : A(3). \quad (3.15)$$

Similarly we get for the other two terms in (3.12)

$$R'_{32} = - : \varphi_2^\dagger(1) D_{m_1}^+(3-1) D_{m_2}^F(3-2) D_{m_3}^+(2-1) \varphi_1(2) : A(3), \quad (3.16)$$

$$R'_{33} = - : \varphi_2^\dagger(1) D_{m_1}^+(3-1) D_{m_2}^+(3-2) D_{m_3}^{AF}(1-2) \varphi_1(2) : A(3). \quad (3.17)$$

The anti-Feynman propagator D^{AF} is the complex conjugate of D^F . It appears because we have used unitary $\tilde{T}_2(1, 2) = T_2(1, 2)^\dagger$. This is the only minor role which unitarity plays here.

The result for A'_3 is obtained in the same way. Collecting the terms with field operators $: \varphi_2^\dagger(1) \varphi_1(2) : A(3)$ in D_3 , the corresponding numerical distribution is given by

$$d_3(1, 2, 3) = -D_{m_1}^F(1-3) D_{m_2}^+(3-2) D_{m_3}^+(1-2) - D_{m_1}^- D_{m_2}^F D_{m_3}^- \quad (3.18)$$

$$+ D_{m_1}^- D_{m_2}^+ D_{m_3}^{AF} + D_{m_1}^F D_{m_2}^- D_{m_3}^- + D_{m_1}^+ D_{m_2}^F D_{m_3}^+ - D_{m_1}^+ D_{m_2}^- D_{m_3}^{AF}, \quad (3.19)$$

where we have used $D^-(x) = -D^+(-x)$ and the arguments in the 6 terms agree with the first term. The 6 terms come from three cuts through the vertex diagram, not only one. To get contact with the convention in [7], we shall use the relative coordinates

$$y_1 = x_1 - x_3, \quad y_2 = x_3 - x_2, \quad (3.20)$$

and calculate the Fourier transform

$$\hat{d}_3(p, q) = (2\pi)^{-4} \int d_3(y_1, y_2) e^{ip y_1 + iq y_2} d^4 y_1 d^4 y_2. \quad (3.21)$$

Then we arrive at

$$\hat{d}_3(p, q) = (2\pi)^{-2} \int dk \left[D_{m_1}^-(p-k) D_{m_2}^+(q-k) D_{m_3}^{AF}(k) \right. \quad (3.22)$$

$$\left. - D_{m_1}^F(p-k) D_{m_2}^+(q-k) D_{m_3}^+(k) - D_{m_1}^-(p-k) D_{m_2}^F(q-k) D_{m_3}^-(k) \right] \\ - [p \leftrightarrow q, m_1 \leftrightarrow m_2]. \quad (3.23)$$

Up to the arbitrary masses and the imaginary parts, this agrees precisely with the result in QED ([7, Eq. (3.8.28)]). By the same techniques as in the QED case, we then find

$$\hat{d}_3(p, q) = \frac{\pi}{4(2\pi)^6} \left\{ \frac{\text{sgn } P_0}{\sqrt{N}} \Theta(P^2 - (m_1 + m_2)^2) \log_1 - \frac{\text{sgn } q_0}{\sqrt{N}} \Theta(q^2 - (m_2 + m_3)^2) \log_2 \right. \\ \left. + \frac{\text{sgn } p_0}{\sqrt{N}} \Theta(p^2 - (m_1 + m_3)^2) \log_3 \right\}, \quad (3.24)$$

where $P = p - q$, $N = (pq)^2 - p^2q^2$ and

$$\log_1 = \log \left| \frac{p^2 + m_1^2 - m_3^2 - pP \left(1 + \frac{m_1^2 - m_2^2}{P^2} \right) + \sqrt{N} \sqrt{1 - 2 \frac{m_1^2 + m_2^2}{P^2} + \left(\frac{m_1^2 - m_2^2}{P^2} \right)^2}}{p^2 + m_1^2 - m_3^2 - pP \left(1 + \frac{m_1^2 - m_2^2}{P^2} \right) - \sqrt{N} \sqrt{1 - 2 \frac{m_1^2 + m_2^2}{P^2} + \left(\frac{m_1^2 - m_2^2}{P^2} \right)^2}} \right|. \quad (3.25)$$

$$\log_2 = \log \left| \frac{p^2 - m_1^2 + m_3^2 - pq \left(1 - \frac{m_2^2 - m_3^2}{q^2} \right) + \sqrt{N} \sqrt{\left(1 - \frac{m_2^2 - m_3^2}{q^2} \right)^2 - \frac{4m_3^2}{q^2}}}{p^2 - m_1^2 + m_3^2 - pq \left(1 - \frac{m_2^2 - m_3^2}{q^2} \right) - \sqrt{N} \sqrt{\left(1 - \frac{m_2^2 - m_3^2}{q^2} \right)^2 - \frac{4m_3^2}{q^2}}} \right|, \quad (3.26)$$

$$\log_3 = \log \left| \frac{q^2 - m_2^2 + m_3^2 - pq \left(1 - \frac{m_1^2 - m_3^2}{p^2} \right) + \sqrt{N} \sqrt{\left(1 - \frac{m_1^2 - m_3^2}{p^2} \right)^2 - \frac{4m_3^2}{p^2}}}{q^2 - m_2^2 + m_3^2 - pq \left(1 - \frac{m_1^2 - m_3^2}{p^2} \right) - \sqrt{N} \sqrt{\left(1 - \frac{m_1^2 - m_3^2}{p^2} \right)^2 - \frac{4m_3^2}{p^2}}} \right|. \quad (3.27)$$

Because of Lemma 2 we only need the real parts of the logarithms. The splitting of (3.24) by means of the central solution (2.52) is done in a later section.

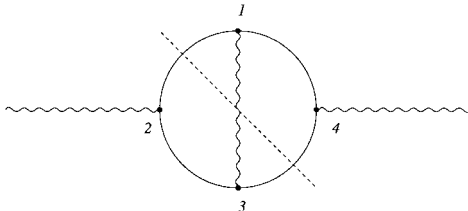


FIG. 3a. The three-particle cut.

3.2. Vacuum Polarization in Fourth Order

3.2.1. The Causal Distribution $d_4(p)$

Now we envisage the calculation of the diagram shown in Fig. 1, which contributes to vacuum polarization in fourth order. We restrict ourselves to the case with vanishing “photon” mass m_3 and mass m of the “electron”. We start from

$$\begin{aligned}
 R'_4 = & T_3(1, 2, 4) \tilde{T}_1(3) + T_3(1, 3, 4) \tilde{T}_1(2) + T_3(2, 3, 4) \tilde{T}_1(1) \\
 & + T_2(1, 4) \tilde{T}_2(2, 3) + T_2(2, 4) \tilde{T}_2(1, 3) \\
 & + T_2(3, 4) \tilde{T}_2(1, 2) + T_1(4) \tilde{T}_3(1, 2, 3).
 \end{aligned} \tag{3.28}$$

Note that we only consider terms with field operator $:A(2)A(4):$. According to Lemma 1, the first, the third and the fifth term in (3.28) vanish in the adiabatic limit. Furthermore, the second term R'_{42} gives the same contribution as R'_{47} , and the same holds true for R'_{44} and R'_{46} .

In x -space, the three-particle contribution R'_{44} (Fig. 3a) is given by

$$R'_{44} = r'_{44}(1, 2, 3, 4) :A(2)A(4):, \tag{3.29}$$

$$r'_{44} = 2iD_m^{AF}(2-3) D_m^F(1-4) D_m^+(1-2) D_m^+(4-3) D_{m_3}^+(1-3). \tag{3.30}$$

The Fourier transform of (3.30) is ($y_i = x_i - x_4$)

$$\begin{aligned}
 r'_{44}(p_1, p_2, p_3) &= (2\pi)^{-6} \int dy_1 dy_2 dy_3 r'_{44}(x_1, x_2, x_3, x_4) e^{ip_1 y_1 + ip_2 y_2 + ip_3 y_3} \\
 &= 2(2\pi)^{-11} \int dq dp' \frac{1}{(p_2 + q)^2 - m^2 - i0} \frac{1}{(p_1 - q - p')^2 - m^2 + i0} \\
 &\quad \times \Theta(q^0) \delta(q^2 - m^2) \Theta(-p_2^0 - p_3^0 - q^0 - p'^0) \\
 &\quad \times \delta((p_2 + p_3 + q + p')^2 - m^2) \Theta(p'^0) \delta(p'^2 - m_3^2),
 \end{aligned} \tag{3.31}$$

which becomes in the adiabatic limit $p_1, p_3 \rightarrow 0$, $p_2 = p$

$$\begin{aligned}
 r'_{44}(p) &= 2(2\pi)^{-11} \int dq dp' \frac{1}{(p+q)^2 - m^2 - i0} \frac{1}{(q-p-p')^2 - m^2 + i0} \\
 &\quad \times \Theta(q_0) \delta(q^2 - m^2) \Theta(p'_0 - q_0) \delta((p' - q)^2 - m^2) \Theta(-p_0 - p'_0) \\
 &\quad \times \delta((p + p')^2 - m_3^2). \tag{3.32}
 \end{aligned}$$

In fact, the calculation of (3.32) has already been performed in [14] by G. Källén and A. Sabry up to terms that vanish for $m_3 \rightarrow 0$. Our calculations confirm their result exactly. Since the calculation is quite instructive from the technical point of view, we describe it here in more detail.

We consider first the integral over q :

$$J(p') = \int dq \Theta(q_0) \delta(q^2 - m^2) \Theta(p'_0 - q_0) \delta((p' - q)^2 - m^2) F(pq, -p'q - pq); \tag{3.33}$$

here we have introduced the abbreviation

$$F(pq, -p'q - pq) = \frac{1}{(p+q)^2 - m^2 - i0} \frac{1}{(q-p-p')^2 - m^2 + i0} \tag{3.34}$$

for the two propagators in (3.32). Since J vanishes if p' is not in the forward light-cone, we choose a special Lorentz frame where $p' = (p'_0, \mathbf{0})$. Then the Jordan-Pauli distributions imply

$$p_0'^2 - 2p'_0 q_0 = 0, \quad p'_0 > 2m, \quad \text{and} \quad |\mathbf{q}|^2 = \frac{p_0'^2}{4} - m^2 = Q^2 \tag{3.35}$$

and we have

$$\begin{aligned}
 J &= \Theta(p'_0) \Theta(p_0'^2 - 4m^2) \frac{Q}{4p'_0} \int d\Omega F(pq, \underbrace{-p'q - pq}_{=\tilde{p}q}) \\
 &= \Theta(p'_0) \Theta(p_0'^2 - 4m^2) \frac{\pi Q}{2p'_0} \int_{-1}^{+1} d \cos \vartheta F, \tag{3.36}
 \end{aligned}$$

where ϑ is the angle between \mathbf{q} and $\tilde{\mathbf{p}}$. Introducing

$$x = \tilde{p}q = \frac{1}{2}p'_0 \tilde{p}_0 - Q |\tilde{\mathbf{p}}| \cos \vartheta, \quad dx = -Q |\tilde{\mathbf{p}}| d \cos \vartheta, \tag{3.37}$$

we can write

$$J = \Theta(p'_0) \Theta(p'^2_0 - 4m^2) \frac{\pi}{2p'_0 |\tilde{\mathbf{p}}|} \int_{x_1}^{x_2} dx F\left(-x - \frac{p'^2_0}{2}, x\right), \quad (3.38)$$

$$x_{1,2} = \frac{1}{2} p'_0 \tilde{p}_0 \mp Q |\tilde{\mathbf{p}}|. \quad (3.39)$$

Using

$$p'_0 |\tilde{\mathbf{p}}| = \sqrt{(p'\tilde{p})^2 - p'^2 \tilde{p}^2}, \quad (3.40)$$

we obtain the following covariant expression for r'_{44} :

$$\begin{aligned} r'_{44}(p) &= \frac{\pi}{(2\pi)^{11}} \int dp' \Theta(-p_0 - p'_0) \delta((p+p')^2 - m_3^2) \\ &\quad \times \frac{\Theta(p'_0) \Theta(p'^2 - 4m^2)}{\sqrt{(p'\tilde{p})^2 - p'^2 \tilde{p}^2}} \int_{x_1}^{x_2} dx F\left(-x - \frac{p'^2}{2}, x\right) \\ &= \frac{\pi}{(2\pi)^{11}} \int d\tilde{p} \Theta(\tilde{p}_0) \delta(\tilde{p}^2 - m_3^2) \\ &\quad \times \frac{\Theta(-\tilde{p}_0 - p_0) \Theta((\tilde{p}+p)^2 - 4m^2)}{\sqrt{(p'\tilde{p})^2 - p'^2 m_3^2}} \int_{x_1}^{x_2} dx F\left(-x - \frac{p'^2}{2}, x\right). \end{aligned} \quad (3.41)$$

Since (3.41) vanishes for p outside the backward light-cone, we set $p = (p_0, \mathbf{0})$

$$\begin{aligned} r'_{44}(p_0) &= \frac{1}{(2\pi)^9} \Theta(-p_0) \Theta(p_0^2 - (2m + m_3)^2) \int_{m_3}^{\infty} \frac{d|\tilde{\mathbf{p}}|}{2E_{p'}} \frac{|\tilde{\mathbf{p}}|^2}{|p_0| |\tilde{\mathbf{p}}|} \int_{x_1}^{x_1} dx F \\ &= \frac{1}{2(2\pi)^{11}} \frac{1}{p^2} \Theta(-p_0) \Theta(p_0^2 - (2m + m_3)^2) \int_{\sqrt{m_3^2 p^2}}^{(p^2/2)\delta_2} dy \int_{x_1}^{x_2} dx F, \end{aligned} \quad (3.42)$$

$$y = |p_0| E_{p'}, \quad \delta_2 = \frac{1}{2} (p^2 - 4m^2 + m_3^2),$$

$$\begin{aligned} x_{1,2} &= \frac{1}{2} \left\{ (y - m_3^2) \mp \sqrt{y^2 - m_3^2 p^2} \sqrt{1 - 4m^2/(p^2 + m_3^2 - 2y)} \right\}, \\ &= \frac{1}{2} (y - m_3^2) \mp \frac{1}{2} \zeta. \end{aligned} \quad (3.43)$$

This can be brought into a more symmetrical form if one chooses $x = \frac{1}{2}(y + z - m_3^2)$:

$$r'_{44}(p) = \frac{1}{4(2\pi)^9} \frac{1}{p^2} \Theta(-p_0) \Theta(p_0^2 - (2m + m_3)^2) \\ \times \int_{\sqrt{m_3^2 p^2}}^{(p^2/2)\delta_2} dy \int_{-\xi}^{+\xi} dz F\left(\frac{y-z-p^2}{2}, \frac{y+z-m_3^2}{2}\right). \quad (3.44)$$

By rescaling $z = \xi z'$, the limits of the z -integral become independent from y :

$$r'_{44}(p) = \frac{1}{4(2\pi)^9} \frac{1}{p^2} \Theta(-p_0) \Theta(p_0^2 - (2m + m_3)^2) B(p), \quad (3.45)$$

$$B = \int_{\sqrt{m_3^2 p^2}}^{(p^2/2)\delta_2} dy \int_{-\xi}^{+\xi} dz \frac{1}{y^2 - z^2} = \int_{\sqrt{m_3^2 p^2}}^{(p^2/2)\delta_2} dy \int_{-1}^{+1} dz' \frac{\xi}{y^2 - \xi^2 z'^2}. \quad (3.46)$$

Interchanging the integrations, we can write

$$B = \int_{-1}^{+1} dz' B_1(z'), \quad (3.47)$$

where

$$B_1(z') = \int_{\sqrt{m_3^2 p^2}}^{(p^2/2)\delta_2} dy \frac{\sqrt{y^2 - m_3^2 p^2} \sqrt{1 - \frac{\delta_1}{1 - y'}}}{y^2 - z'^2 (y'^2 - m_3^2 p^2) \left(1 - \frac{\delta_1}{1 - y'}\right)} \quad (3.48)$$

$$= \int_{\varepsilon}^{\delta_3} dy' \frac{\sqrt{y'^2 - \varepsilon^2} \sqrt{1 - \frac{\delta_1}{1 - y'}}}{y'^2 - z'^2 (y'^2 - \varepsilon^2) \left(1 - \frac{\delta_1}{1 - y'}\right)}, \quad (3.49)$$

$$y' = \frac{2}{p^2 + m_3^2} y, \quad \varepsilon = \frac{4m_3^2 p^2}{(p^2 + m_3^2)^2}, \\ \delta_1 = \frac{4m^2}{p^2 + m_3^2}, \quad \delta_3 = \frac{p^2 - 4m^2 + m_3^2}{p^2 + m_3^2}. \quad (3.50)$$

It would be quite difficult to solve this integral. Fortunately, B_1 can be decomposed into two parts for small ε , whereas the second part contains the infrared divergence and the first one is finite for $\varepsilon \rightarrow 0$:

$$B_1(z) = B_{11}(z) + B_{12}(z), \quad (3.51)$$

$$B_{11}(z) = \int_0^{\delta_3} \frac{dy}{y} \left\{ \frac{\sqrt{1 - \frac{\delta_1}{1-y}}}{1 - z^2 \left(1 - \frac{\delta_1}{1-y}\right)} - \frac{\sqrt{1 - \delta_1}}{1 - z^2(1 - \delta_1)} \right\}, \quad (3.52)$$

$$B_{12}(z) = \int_\varepsilon^{\delta_3} dy \frac{\sqrt{y^2 - \varepsilon^2} \sqrt{1 - \delta_1}}{y^2 - z^2(y^2 - \varepsilon^2)(1 - \delta_1)}. \quad (3.53)$$

The calculation of B_{11} can be performed without difficulty if we set

$$t^2 = 1 - \frac{\delta_1}{1-y}. \quad (3.54)$$

The result is (always neglecting terms which will vanish in the limit $\varepsilon \rightarrow 0$)

$$B_{11}(z) = \frac{1}{1-z^2} \log \frac{1-\delta}{1+\delta} + \frac{\delta}{1-\delta^2 z^2} \log \frac{4}{1-\delta^2} + \frac{z(1-\delta^2)}{(1-z^2)(1-\delta^2 z^2)} \log \frac{1+\delta z}{1-\delta z}, \quad \delta = \sqrt{1-4m^2/p^2}. \quad (3.55)$$

In B_{12} we set

$$t^2 = 1 - \varepsilon^2/y^2 \quad (3.56)$$

and obtain

$$B_{12}(z) = \frac{\delta}{2} \frac{1}{1-\delta^2 z^2} \left\{ \log \frac{4\delta^4}{\varepsilon^2} + \frac{1}{\delta z} \log \frac{1-\delta z}{1+\delta z} \right\}. \quad (3.57)$$

The sum is

$$B_1(z) = \frac{1}{1-z^2} \log \frac{1-\delta}{1+\delta} + \frac{\delta}{1-\delta^2 z^2} \log \frac{8\delta^2}{\varepsilon(1-\delta^2)} + \left[\frac{z(1-\delta^2)}{(1-z^2)(1-\delta^2 z^2)} - \frac{1}{2z(1-\delta^2 z^2)} \right] \log \frac{1+\delta z}{1-\delta z}. \quad (3.58)$$

The last integration introduces the dilogarithm, defined for $z \notin [1, \infty)$ by

$$\text{Li}(z) = - \int_0^z dt \frac{\log(1-t)}{t}. \quad (3.59)$$

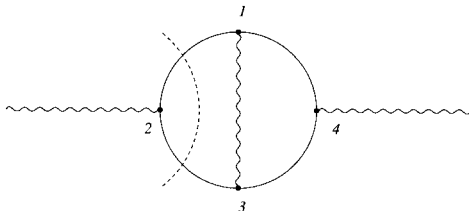


FIG. 3b. The two-particle cut

With the aid of the formulae given in Appendix B we can write the result as

$$B = 3\text{Li}\left(\frac{1-\delta}{1+\delta}\right) + 2\text{Li}\left(-\frac{1-\delta}{1+\delta}\right) - \frac{\pi^2}{3} + \log\frac{1+\delta}{1-\delta}\left[\log\frac{m}{m_3} + \frac{1}{4}\log\frac{1+\delta}{1-\delta} + \log\frac{(1+\delta)^2}{4\delta}\right], \quad (3.60)$$

or

$$B(z) = 3\text{Li}_2(z) + 2\text{Li}_2(-z) + \log z \log(1-z) + \log z \log(1+z) + \frac{1}{4}\log^2 z - \frac{\pi^2}{3} + \log z \log\frac{m_3}{m}. \quad (3.61)$$

Here we have introduced the variable

$$z = \frac{1 - \sqrt{1 - 4m^2/p^2}}{1 + \sqrt{1 - 4m^2/p^2}}, \quad (3.62)$$

which varies from 0 to 1 for $p^2 \in [4m^2, \infty)$.

The two-particle contribution r'_{42} (Fig. 3b) is

$$r'_{42}(p) = 2i(2\pi)^{-2} \int dp' D_m^+(-p') A_3(p', p' - p) D_m^+(p' - p). \quad (3.63)$$

Here, A_3 is the vertex function. Since its first argument lies in the backward light-cone and the second one in the forward light-cone, we are in the case of lower signs in (2.52), because of our convention (3.21). Then the retarded vertex function A_3^{ret} is given by the dispersion integral

$$A_3^{ret}(p, q) = -\frac{i}{2\pi} \int_{-\infty}^{+\infty} dt \frac{d_3(tp, tq)}{1-t-i0}. \quad (3.64)$$

Again, the computation of (the real part of) r'_{42} can be carried out in a straightforward manner, as we shall see below. The vertex function consists of three different parts. The first one leads to an infrared finite contribution to r'_{42} ,

$$r'_{42}(p) = \frac{1}{4(2\pi)^9} \frac{1}{p^2} \Theta(-p_0) \Theta(p^2 - 4m^2) C_1(z), \quad (3.65)$$

$$C_1(z) = 2\text{Li}_2(z) + 2 \log z \log(1-z) - \frac{1}{2} \log^2 z + \frac{\pi^2}{6}, \quad (3.66)$$

whereas the second and third term are equal and contain an infrared divergent term $\sim \log(m_3/m)$, which cancels the infrared divergence in (3.61):

$$r'_{42}(p) = \frac{1}{4(2\pi)^9} \frac{1}{p^2} \Theta(-p_0) \Theta(p^2 - 4m^2) C_2(z), \quad (3.67)$$

$$C_2(z) = -\text{Li}_2(z) - \log z \log(1-z) + \frac{1}{4} \log^2 z + \frac{\pi^2}{6} - \log z \log \frac{m_3}{m}. \quad (3.68)$$

We turn to the calculation of the term associated with the first part of the vertex function

$$\begin{aligned} r'_{42}(p) &= -\frac{\pi}{2(2\pi)^9} \int_{-\infty}^{+\infty} dt \frac{\text{sgn } t \Theta(t^2 p^2 - 4m^2)}{t^2(1-t+i0)} \\ &\times \int dp' \left(\frac{i}{2\pi} \right)^2 \Theta(-p'_0) \delta(p'^2 - m^2) \Theta(p'_0 - p_0) \delta((p' - p)^2 - m^2) \\ &\times \frac{1}{\sqrt{N_{p,p'}}} \log \left| \frac{t^2(p'^2 - pp') + m^2 - m_3^2 + t^2 \sqrt{N_{p,p'}} \sqrt{1 - 4m^2/p^2 t^2}}{t^2(p'^2 - pp') + m^2 - m_3^2 - t^2 \sqrt{N_{p,p'}} \sqrt{1 - 4m^2/p^2 t^2}} \right|. \end{aligned} \quad (3.69)$$

Since we only have to calculate the causal distribution d_4 for p in the light-cone, we can choose a Lorentz frame where $p = (p_0, \mathbf{0})$, $p_0 < 0$:

$$\begin{aligned} r'_{42}(p_0) &= \frac{\pi}{2(2\pi)^{11}} \Theta(-p_0) \int_{t^2 > 4m^2/p_0^2} dt \frac{\text{sgn } t}{t^2(1-t)} \\ &\times \int \frac{d^3 p'}{2E'} \underbrace{\frac{1}{|p_0| |\mathbf{p}'|}}_{\sqrt{N_{p,p'}}} \frac{\delta(E' - |p_0/2|)}{2 |p_0|} \log_1 \\ &= \frac{1}{8(2\pi)^9} \Theta(-p_0) \Theta(p_0^2 - 4m^2) \frac{1}{p_0^2} \int_{t^2 > 4m^2/p_0^2} dt \frac{\text{sgn } t}{t^2(1-t)} \\ &\times \log \left| \frac{t^2(m^2 - p_0^2/2) + m^2 - m_3^2 + t^2 \sqrt{p_0^2/4 - m^2} \sqrt{p_0^2 - 4m^2/t^2}}{t^2(m^2 - p_0^2/2) + m^2 - m_3^2 - t^2 \sqrt{p_0^2/4 - m^2} \sqrt{p_0^2 - 4m^2/t^2}} \right|. \end{aligned} \quad (3.70)$$

The generalization to arbitrary p is obvious. The computation of the integral

$$\begin{aligned}
 J &= \int_{t^2 > 4m^2/p^2} dt \frac{\text{sgn } t}{t^2(1-t)} \log_1 = \int_{\sqrt{4b}}^{\infty} dt \left(\frac{1}{1-t} - \frac{1}{1+t} \right) \frac{\log_1}{t^2} \\
 &= \int_{4b}^{\infty} \frac{ds}{s(1-s)} \log \left| \frac{s(m^2 - p^2/2) + m^2 - m_3^2 + s\sqrt{p^2/4 - m^2}\sqrt{p^2 - 4m^2/s}}{s(m^2 - p^2/2) + m^2 - m_3^2 - s\sqrt{p^2/4 - m^2}\sqrt{p^2 - 4m^2/s}} \right|, \quad (3.71)
 \end{aligned}$$

where $b = m^2/p^2$, is straightforward: We substitute

$$\sqrt{s^2 p^2 - 4m^2 s} = xs, \quad s = \frac{4m^2}{p^2 - x^2}, \quad \frac{ds}{s} = \frac{2x}{p^2 - x^2} dx \quad (3.72)$$

and obtain

$$\begin{aligned}
 J &= \int_0^{\sqrt{p^2}} dx \frac{2x}{p^2 - x^2 - 4m^2} \\
 &\quad \times \log \left| \frac{(1 - m_3^2/m^2)x^2 + (1 + m_3^2/m^2)p^2 - 4m^2 - 4x\sqrt{p^2/4 - m^2}}{(1 - m_3^2/m^2)x^2 + (1 + m_3^2/m^2)p^2 - 4m^2 + 4x\sqrt{p^2/4 - m^2}} \right|. \quad (3.73)
 \end{aligned}$$

Since there appears no infrared divergence in this expression, we can set $m_3 = 0$:

$$\begin{aligned}
 J &= \int_0^{\sqrt{p^2}} dx \frac{2x}{p^2 - x^2 - 4m^2} \log \frac{x^2 - 2ax + a^2}{x^2 + 2ax + a^2} \\
 &= 2 \int_0^1 dx \left(\frac{1}{x + \delta} + \frac{1}{x - \delta} \right) \log \left| \frac{x + \delta}{x - \delta} \right|, \quad \delta = \sqrt{1 - 4m^2/p^2}. \quad (3.74)
 \end{aligned}$$

Using the identities given in Appendix B we obtain the final result

$$\begin{aligned}
 J &= 4\text{Li} \left(\frac{1 - \delta}{1 + \delta} \right) + \log \left(\frac{(1 + \delta)^2}{4\delta^2} \right) \log \left(\frac{1 + \delta}{1 - \delta} \right) \\
 &\quad + \log \left(\frac{1 - \delta^2}{4\delta^2} \right) \log \left(\frac{1 + \delta}{1 - \delta} \right) + \frac{\pi^2}{3}. \quad (3.75)
 \end{aligned}$$

The calculation of the second and third contribution to r'_{42} leads to the integral

$$I = \int_{2mm_3}^{\infty} dy \left(\frac{1}{y + m^2 + m_3^2} - \frac{1}{y + m_3^2} \right) \log \left| \frac{\frac{p^2}{2} y + \sqrt{N} \sqrt{y + p^2 m_3^2}}{\frac{1}{p^2} y - \sqrt{N} \sqrt{y + p^2 m_3^2}} \right|, \quad (3.76)$$

$$N = \frac{p^4}{4} - m^2 p^2, \quad \sqrt{y} = \sqrt{y^2 - 4m^2 m_3^2}. \quad (3.77)$$

Introducing $(y^2 - 4m^2m_3^2)^2 = (y-t)^2$ enables us to represent the integral in the following way ($z = (3.62)$)

$$I = \lim_{\varepsilon \rightarrow 0} (I_1^\varepsilon + I_2^\varepsilon), \quad (3.78)$$

$$I_1^\varepsilon = \int_\varepsilon^1 dt \frac{t^2 - 1}{t} \frac{t^2 - 1}{t^2 + \frac{m_3}{m}t + 1} \log \left| \frac{t^2 + \frac{m_3}{m} \frac{1+z}{z} t + 1/z}{\frac{t^2}{z} + \frac{m_3}{m} \frac{1+z}{z} t + 1} \right|, \quad (3.79)$$

$$I_2^\varepsilon = - \int_\varepsilon^1 dt \frac{t^2 - 1}{t} \frac{t^2 - 1}{t^2 + \frac{m^2 + m_3^2}{mm_3}t + 1} \log \left| \frac{t^2 + \frac{m_3}{m} \frac{1+z}{z} t + 1/z}{\frac{t^2}{z} + \frac{m_3}{m} \frac{1+z}{z} t + 1} \right|. \quad (3.80)$$

In addition we decompose carefully the expressions above into partial fractions

$$I_1^\varepsilon = \int_\varepsilon^1 dt \left(\frac{2t}{t^2 + 1} - \frac{1}{2} \right) \log \left| \frac{t^2 + 1/z}{t^2/z + 1} \right| + o(m_3), \quad (3.81)$$

$$I_2^\varepsilon = - \int_\varepsilon^1 dt \left(\frac{t_1^2 - 1}{t_1(t_1 - t_2)} \frac{1}{t - t_1} + \frac{t_2^2 - 1}{t_2(t_2 - t_1)} \frac{1}{t - t_2} - \frac{1}{t} \right) \log \left| \frac{t^2 + 1/z}{t^2/z + 1} \right| + o(m_3), \quad (3.82)$$

$$t_{1,2} = -\frac{m}{m_3}, -\frac{m_3}{m}, \quad t_1 t_2 = 1. \quad (3.83)$$

Hence, we have to evaluate

$$I = \tilde{I}_1 + \tilde{I}_2 = \int_0^1 dt \frac{2t}{t^2 + 1} \log \left| \frac{t^2 + 1/z}{t^2/z + 1} \right| - \int_0^1 dt \frac{1}{t + (m_3/m)} \log \left| \frac{t^2 + 1/z}{t^2/z + 1} \right|, \quad (3.84)$$

again ignoring terms which vanish for $m_3 \rightarrow 0$. It is advantageous not to calculate directly the integrals obtained so far, but the derivative of I with respect to z

$$\frac{d}{dz} I = \frac{1}{1-z} \log z + \frac{1}{2z} \log z - \frac{1}{z} \log \frac{m_3}{m}, \quad (3.85)$$

and this yields, since $I(z=1) = 0$, the desired result

$$I = -\text{Li } z - \log z \log(1-z) + \frac{1}{4} \log^2 z - \log z \log \frac{m_3}{m} + \frac{\pi^2}{6}. \quad (3.86)$$

Now $d_4 = r'_4 - a'_4$ can immediately be written down if we note that

$$r'_{4i}(p) = a'_{4i}(-p), \quad i = 1, \dots, 7. \quad (3.87)$$

3.2.2. The Splitting of $d_4(p)$

After having calculated the causal distribution d_4 ,

$$d_4(p) = -\frac{1}{2(2\pi)^9} \frac{1}{p^2} \operatorname{sgn} p_0 \Theta(p^2 - 4m^2) J(z), \quad (3.88)$$

$$J(z) = 4\operatorname{Li}_2(z) + 2\operatorname{Li}_2(-z) + 2\log(z)\log(1-z) + \log(z)\log(1+z),$$

we must decompose it into retarded and advanced parts. The retarded distribution r_4 is given in the forward light-cone according to (2.52)

$$\begin{aligned} r_4(p) &= \frac{i}{2\pi} \int_{-\infty}^{+\infty} dt \frac{d_4(tp)}{1-t+i0} \\ &= -\frac{i}{2\pi} \frac{1}{2(2\pi)^9} \frac{1}{p^2} \int_{-\infty}^{+\infty} \frac{\operatorname{sgn} t \Theta(t^2 p^2 - 4m^2)}{t^2(1-t+i0)} J\left(\frac{\sqrt{t^2 p^2} - \sqrt{t^2 p^2 - 4m^2}}{\sqrt{t^2 p^2} + \sqrt{t^2 p^2 - 4m^2}}\right) dt. \end{aligned} \quad (3.89)$$

It is very convenient to introduce the new integration variable

$$\begin{aligned} x &= \frac{\sqrt{t^2 p^2} - \sqrt{t^2 p^2 - 4m^2}}{\sqrt{t^2 p^2} + \sqrt{t^2 p^2 - 4m^2}}, & t &= \frac{1}{\sqrt{b}} \frac{x+1}{\sqrt{x}}, \\ b &= p^2/m^2, & \frac{dt}{dx} &= \frac{1}{\sqrt{b}} \frac{x-1}{2x\sqrt{x}}. \end{aligned} \quad (3.90)$$

This leads to ($p \in V^+$)

$$r_4(p) = -\frac{i}{2(2\pi)^{10}} \frac{1}{p^2} \int_0^1 dx \left\{ -\frac{2}{x+1} + \frac{1}{x-z} + \frac{1}{x-1/z} \right\} J(x). \quad (3.91)$$

The imaginary part $i0$ in the denominator of (3.89) is included in p^2 as discussed at the end of this section.

From now on, $r_4(p)$ is considered as a function of z (3.62),

$$z = \frac{1}{2} \left(b - 2 - b \sqrt{1 - 4/b} \right) = \frac{1 - \sqrt{1 - 4m^2/p^2}}{1 + \sqrt{1 - 4m^2/p^2}}. \quad (3.92)$$

Then we note that $z \in [0, 1]$ for $p^2 \in [4m^2, \infty)$, and

$$R(z) = \int_0^1 dx \left\{ -\frac{2}{x+1} + \frac{1}{x-1/z} + \frac{1}{x-z} \right\} J(x) \quad (3.93)$$

has the property $R(z) = R(1/z)$. The integral

$$R_1 = -2 \int_0^1 \frac{dx}{x+1} J(x) = -\frac{9}{4} \zeta(3) \quad (3.94)$$

is just a constant and can be calculated with the formulae given in [21, 22]. The calculation of

$$R_2(z) = \int_0^1 \frac{J(x)}{x-z} + \int_0^1 \frac{J(x)}{x-1/z} = R_3(z) + R_4(z) \quad (3.95)$$

is most easily performed by first calculating the derivatives of R_3 and R_4 :

$$\begin{aligned} R_3'(z) &= \partial_z \left\{ \frac{1}{z} \int_0^1 dx \frac{J(x)}{x/z-1} \right\} = \partial_z \left\{ \int_0^{1/z} dt \frac{J(tz)}{t-1} \right\} \\ &= \frac{J(1)}{z(z-1)} + \frac{1}{z} \int_0^1 dx \frac{x}{x-z} J'(x), \end{aligned} \quad (3.96)$$

$$R_4'(z) = \frac{J(1)}{z-1} + \int_0^1 dx \frac{J'(x)}{1/x-z}. \quad (3.97)$$

This is a helpful trick, but all integrals coming up in (3.95) are also discussed in the literature mentioned above. We give the separate results for the two-particle and three-particle expressions

$$\begin{aligned} & \int_0^1 dx \left\{ -\frac{2}{x+1} + \frac{1}{x-1/z} + \frac{1}{x-z} \right\} B(x) \\ &= 5\text{Li}_3(z) + 3\text{Li}(-z) - 3\text{Li}(z) \log z - 2\text{Li}(-z) \log z - \frac{1}{2} \log^2 z \log(1-z) \\ & \quad - \frac{1}{2} \log^2 z \log(1+z) - \frac{1}{12} \log^3 z + \frac{\pi^2}{2} \log(1+z) + \frac{\pi^2}{2} \log z \\ & \quad + \frac{3}{4} \zeta(3) + \frac{\pi^2}{2} \log 2, \end{aligned} \quad (3.98)$$

and

$$\begin{aligned} & \int_0^1 dx \left\{ -\frac{2}{x+1} + \frac{1}{x-1/z} + \frac{1}{x-z} \right\} C(x) \\ &= \text{Li}_3(z) - \text{Li}(z) \log z - \frac{1}{2} \log^2 z \log(1-z) + \frac{1}{12} \log^3 z \\ & \quad + \pi^2 \log(1-z) - \frac{\pi^2}{2} \log z + \frac{3}{4} \zeta(3) - \frac{\pi^2}{2} \log 2, \end{aligned} \quad (3.99)$$

where $C(x) = C_1(x) + C_2(x)$. Here we have introduced the trilogarithm, which is defined by

$$\text{Li}_3(z) = \int_0^z dx \frac{\text{Li}(x)}{x}. \quad (3.100)$$

The infrared divergent terms in B and C which cancel mutually in r_4 have already been omitted. Finally r_4 is given by

$$r_4(z) = -\frac{i}{2(2\pi)^{10} p^2} R(z), \quad (3.101)$$

$$\begin{aligned} R(z) = & 6\text{Li}_3(z) + 3\text{Li}_3(-z) - 4\text{Li}_2(z) \log(-z) - 2\text{Li}_2(-z) \log(-z) \\ & - \log^2(-z) \log(1-z) - \frac{1}{2} \log^2(-z) \log(1+z) + \frac{3}{2} \zeta(3). \end{aligned} \quad (3.102)$$

If $R(z)$ becomes complex the sign of the imaginary parts is determined by the $i0$ in (3.89) as follows. For arbitrary time-like p , the $i0$ goes over into $i0p_0$ (see 2.52) and [7, Chap. 3.6]). To obtain the fulltime-ordered distribution $t_4(p)$ we have to subtract $r'_4(p)$. This changes $i0p_0$ into $i0$ again. Consequently, $t_4(p)$ is given by (3.101) with p^2 substituted by $p^2 + i0$, i.e. $z \rightarrow z + i0$. This fixes the signs of the imaginary parts at the logarithmic cuts in (3.102). For space-like p , $t_4(p)$ is simply given by (3.101) because $r'_4(p)$ now vanishes and $R(z)$ is real.

The two difficult integrations in the vacuum polarization are the two dispersion integrals in the two-particle contribution, and one phase space integration and one dispersion integral in the three-particle contribution.

3.3. The Electron Propagator in Fourth Order

After having demonstrated the instructive example of vacuum polarization, we proceed now with the discussion of the electron propagator in fourth order. Again, we start from the general expression (3.28) and consider only terms with field operator $:\varphi(2)\varphi^\dagger(4):$. Just as in the case of vacuum polarization, the first, the third and the fifth term in (3.28) vanish in the adiabatic limit, and the second and the seventh term

$$R'_{42} = T_3(1, 3, 4) \tilde{T}_1(2), \quad R'_{47} = T_1(4) \tilde{T}_3(1, 2, 3) \quad (3.103)$$

give the same contribution to the propagator. But we have to distinguish the different cuts in R'_{44} and R'_{46} .

In x -space, the three-particle contribution in R'_{46} with two photons and one electron as intermediate state (Fig. 4) is given by

$$R'_{46} = r'_{46}(1, 2, 3, 4) : \varphi(2)\varphi^\dagger(4) :, \quad (3.104)$$

$$r'_{46} = iD_m^F(3-4) D_m^{AF}(1-2) D_m^+(3-1) D_{m_3}^+(4-1) D_{m_3}^+(3-2). \quad (3.105)$$

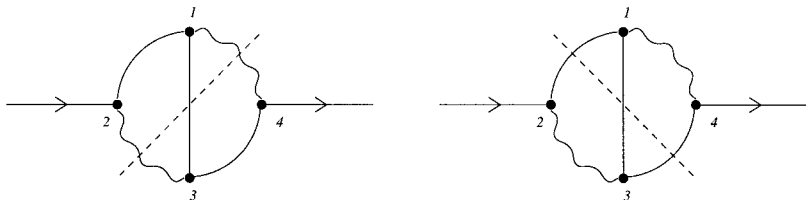


FIG. 4. The two different three-particle cuts.

The Fourier transform is after the adiabatic limit

$$\begin{aligned}
 r'_{46}(p) &= (2\pi)^{-11} \int dq dp' \frac{1}{(p+q)^2 - m^2 - i0} \frac{1}{(q-p-p')^2 - m^2 + i0} \\
 &\quad \times \Theta(q_0) \delta(q^2 - m_3^2) \Theta(p'_0 - q_0) \delta((p' - q)^2 - m_3^2) \\
 &\quad \times \Theta(-p_0 - p'_0) \delta((p+p')^2 - m^2). \tag{3.106}
 \end{aligned}$$

Straightforward calculation of (3.106) with the same technique as in the case of vacuum polarization leads to the simple result

$$r'_{46} = \frac{1}{8(2\pi)^9} \frac{1}{p^2} \Theta(-p_0) \Theta(p^2 - m^2) B(x^2), \tag{3.107}$$

$$B(x^2) = \frac{1}{2} \text{Li}_2(x^2) + \frac{1}{2} \log(x^2) \log(x^2 - 1) - \frac{\pi^2}{12}, \quad x^2 = p^2/m^2. \tag{3.108}$$

The two-particle contribution can be evaluated without any problem with the aid of (3.124). We therefore quote only the result:

$$R'_{42} = r'_{42}(1, 2, 3, 4) : \varphi(2) \varphi^\dagger(4);, \tag{3.109}$$

$$r'_{42}(p) = \frac{1}{4(2\pi)^9} \frac{1}{p^2} \Theta(-p_0) \Theta(p^2 - m^2) C(x^2), \tag{3.110}$$

$$C(x^2) = \frac{1}{2} \text{Li}_2(x^2) - \frac{\pi^2}{12}. \tag{3.111}$$

In (3.108) and (3.111) only the real parts of the dilogarithms contribute.

Since we have to treat the three-particle contribution with three electrons as intermediate state (r'_{44} and r'_{46}) separately, we already give the splitting results for the expressions obtained so far. The splitting procedure leads to the same type of

integrals as in the case of vacuum polarization, hence we refer again to [21, 22]. We have for p in the forward light-cone and $p^2 > m^2$

$$d_1(p) = -\frac{1}{8(2\pi)^9} \frac{1}{p^2} \operatorname{sgn} p_0 \Theta(p^2 - m^2) \left[3\operatorname{Li}_2(x^2) + \log x^2 \log(x^2 - 1) - \frac{\pi^2}{2} \right], \quad (3.112)$$

$$r_1(p) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dt \frac{d_1(tp)}{1-t+i0} \quad (3.113)$$

$$\begin{aligned} &= -\frac{i}{8(2\pi)^{10}} \frac{1}{p^2} [4\operatorname{Li}_3(1-x^2) - 3 \log(1-x^2) \operatorname{Li}_2(1-x^2) \\ &\quad - \log x^2 \log^2(1-x^2) - 4\zeta(3)]. \end{aligned} \quad (3.114)$$

The calculation of the second diagram in Fig. 4

$$\begin{aligned} r'_{44}(p) &= (2\pi)^{-11} \int dq dp' \frac{1}{(p+q)^2 - m_3^2 - i0} \frac{1}{(q-p-p')^2 - m_3^2 + i0} \\ &\quad \times \Theta(q_0) \delta(q^2 - m^2) \Theta(p'_0 - q_0) \delta((p' - q)^2 - m^2) \\ &\quad \times \Theta(-p_0 - p'_0) \delta((p+p')^2 - m^2) \end{aligned} \quad (3.115)$$

is difficult and requires extra consideration. $r'_{44}(p)$ turns out to be infrared finite, so we may drop the small photon mass in the following. Making use of all δ -distributions, the integral

$$I = \int dq \Theta(q_0) \delta(q^2 - m^2) \Theta(p'_0 - q_0) \delta((p' - q)^2 - m^2) \frac{1}{(q+p)^2 - i0} \frac{1}{(q-p'-p)^2 + i0} \quad (3.116)$$

can be transformed into

$$I = -\frac{\pi}{2} \frac{\Theta(p'_0) \Theta(p'^2 - 4m^2)}{\sqrt{(p'\tilde{p})^2 - p'^2\tilde{p}^2}} \int_{x_1}^{x_2} dx \frac{1}{4(x+m^2)(x+p\tilde{p})}, \quad (3.117)$$

where

$$\tilde{p} = -p - p', \quad x_{1,2} = \frac{1}{2} p' \tilde{p} \mp \frac{1}{2} \sqrt{(p'\tilde{p})^2 - p'^2\tilde{p}^2} \sqrt{1 - \frac{4m^2}{p'^2}}. \quad (3.118)$$

Then we obtain

$$r'^2_{44}(p) = -\frac{1}{4(2\pi)^9} \frac{1}{p^2} \Theta(-p_0) \Theta(p^2 - 9m^2) \int_{\sqrt{p^2 m^2}}^{(1/2)(p^2 - 3m^2)} dy \\ \times \int_{x_1}^{x_2} dx \frac{1}{4(x+m^2)(x-y)}, \quad (3.119)$$

$$x_{1,2} = \frac{1}{2}(y - m^2) \mp \frac{\xi}{2}, \quad \xi = (y^2 - p^2 m^2) \sqrt{1 - \frac{4m^2}{p^2 + m^2 - 2y}}. \quad (3.120)$$

Introducing $x = (y + z - m^2)/2$, $d_2(p) = 2(r'^2_{44}(p) - a'^2_{44}(p))$ becomes

$$d_2(p) = -\frac{1}{4(2\pi)^9} \frac{1}{p^2} \operatorname{sgn} p_0 \Theta(p^2 - 9m^2) \int_{\sqrt{p^2 m^2}}^{(1/2)(p^2 - 3m^2)} dy \\ \times \int_{-\xi}^{+\xi} dz \frac{1}{(y+z+m^2)(y-z+m^2)}. \quad (3.121)$$

At this stage, we will not proceed the same way as in the case of vacuum polarization. We first apply the splitting formula to (3.121)

$$r_2(p) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} dt \frac{d_2^2(p)}{1-t+i0} \\ = -\frac{i}{4(2\pi)^{10}} \frac{1}{p^2} \int_{9(m^2/p^2)}^{+\infty} \frac{ds}{s(1-s+i0)} \int_{\sqrt{p^2 m^2 s}}^{(1/2)(p^2 s - 3m^2)} dy \\ \times \int_{-\xi(s,y)}^{+\xi(s,y)} dz \frac{1}{(y+z+m^2)(y-z+m^2)}, \quad (3.122)$$

$$\xi(s, y) = \sqrt{y^2 - p^2 m^2 s} \sqrt{1 + \frac{4m^2}{2y - m^2 - p^2 s}}, \quad p \in V^+ \quad (3.123)$$

and perform a partial integration with respect to s . This leads to the integral

$$r_2(p) = \frac{i}{2(2\pi)^{10}} \frac{1}{p^2} \int_{9(m^2/p^2)}^{+\infty} ds \log\left(\frac{s}{1-s-i0}\right) \int_{\sqrt{p^2 m^2 s}}^{(1/2)(p^2 s - 3m^2)} dy \\ \times \frac{\frac{d}{ds} \xi(s, y)}{(y+m^2)^2 - \xi^2(s, y)} \quad (3.124)$$

$$= \frac{i}{2(2\pi)^{10}} \frac{1}{p^2} \int_{9/x^2}^{+\infty} ds \log\left(\frac{s}{1-s-i0}\right) \int_{\sqrt{x^2 s}}^{y_2} dy \frac{\xi'}{(y+1)^2 - \xi^2}, \quad (3.125)$$

$$\begin{aligned}
 x^2 &= p^2/m^2, & y_1 &= \frac{1}{2}(x^2s + 1), & y_2 &= \frac{1}{2}(x^2s - 3), \\
 \xi &= \sqrt{y^2 - x^2s} \sqrt{\frac{y_2 - y}{y_1 - y}}.
 \end{aligned} \tag{3.126}$$

We obtain after some simple manipulations

$$\begin{aligned}
 r_2 &= \frac{i}{2(2\pi)^{10} m^2} \int_{9/x^2}^{\infty} ds \frac{\log\left(\frac{s}{1-s-i0}\right)}{(x^2s-1)^2} \\
 &\times \int_{\sqrt{x^2s}}^{y_2} \frac{dy}{\sqrt{(y^2-x^2s)(y-y_1)(y-y_2)}} \left[(x^2s-1)y - \frac{1}{4}(x^2s+1)^2 + 1 \right].
 \end{aligned} \tag{3.127}$$

The last integral can be expressed by complete elliptic integrals of the first and third kind. It is

$$\int_{\sqrt{x^2s}}^{y_2} \frac{dy}{\sqrt{(y^2-x^2s)(y_1-y)(y_2-y)}} = \frac{2}{\sqrt{(y_1-\sqrt{x^2s})(y_2+\sqrt{x^2s})}} K(k), \tag{3.128}$$

where

$$k^2 = \frac{(y_2 - \sqrt{x^2s})(y_1 + \sqrt{x^2s})}{(y_1 - \sqrt{x^2s})(y_2 + \sqrt{x^2s})}, \tag{3.129}$$

and

$$\begin{aligned}
 &\int_{\sqrt{x^2s}}^{y_2} \frac{y dy}{\sqrt{(y^2-x^2s)(y_1-y)(y_2-y)}} \\
 &= \frac{2}{\sqrt{(y_1-\sqrt{x^2s})(y_2+\sqrt{x^2s})}} [2\sqrt{x^2s} \Pi(\alpha^2, k) - \sqrt{x^2s} K(k)],
 \end{aligned} \tag{3.130}$$

$$\alpha^2 = \frac{y_2 - \sqrt{x^2s}}{y_2 + \sqrt{x^2s}}. \tag{3.131}$$

Introducing $\lambda = \sqrt{x^2s} \in [3, \infty)$, we find that the modulus k and the parameter α^2 are related by the identities

$$\alpha^2 = \frac{(\lambda-3)(\lambda+1)}{(\lambda+3)(\lambda-1)}, \quad k^2 = \frac{(\lambda-3)(\lambda+1)^3}{(\lambda+3)(\lambda-1)^3}. \tag{3.132}$$

Then it is in fact possible to express $\Pi(\alpha^2, k)$ by $K(k)$:

$$\Pi(\alpha^2, k) = \frac{\lambda + 3}{6} K(k). \quad (3.133)$$

This has already been observed by A. Sabry [23]. Our relation follows from his equation (85) by the substitution $\lambda \rightarrow 1/\lambda$. Finally we arrive at a remarkably simple expression for r_2 :

$$r_2(x) = \frac{i}{6(2\pi)^{10} m^2} \int_{9/x^2}^{\infty} ds \frac{\lambda - 3}{(\lambda - 1)^2 (\lambda + 1)} \sqrt{\frac{\lambda + 3}{\lambda - 1}} \log\left(\frac{s}{1 - s - i0}\right) K(k). \quad (3.134)$$

So far we have calculated the retarded distribution for time-like $x^2 = p^2/m^2$. By the same argument as given at the end of the last section, this also gives the time-ordered distribution $t_2(p)$ for arbitrary p^2 . To get the full fermion propagator, one must add $t_1(p)$ given by (3.114) with $x^2 \rightarrow x^2 + i0$. The integral in (3.134), which can easily be computed numerically, is in agreement with Eq. (30) of Broadhurst [15]. Our funny normalization factors are the correct ones for the calculation of S -matrix elements according to (1.1). In order to permit direct comparison to Feynman integral calculations, we give the following threshold values for $p^2 = m^2$:

$$\begin{aligned} t_1(m^2) &= \frac{i}{2(2\pi)^{10} m^2} \zeta(3), \\ t_2(m^2) &= \frac{i}{8(2\pi)^{10} m^2} \left(\pi^2 \log 2 - \frac{11}{2} \zeta(3) \right), \\ t &= t_1 + t_2 = \frac{i}{8(2\pi)^{10}} \left(\pi^2 \log 2 - \frac{3}{2} \zeta(3) \right), \end{aligned} \quad (3.135)$$

and

$$I(m^2) = \pi^2 \log 2 - \frac{3}{2} \zeta(3); \quad (3.136)$$

here

$$I(p^2) = \frac{p^2}{\pi^4} \int d^4k \int d^4k' \frac{1}{k'^2(k-p)^2(k^2-m^2)((k-k')^2-m^2)((k'-p)^2-m^2)}. \quad (3.137)$$

We have again encountered two difficult integrations: the first one in (3.127) leads to elliptic integrals and the second one (3.134) cannot be expressed by known special functions.

4. THE REMAINING TWO-LOOP DIAGRAMS

4.1. Vacuum Polarization

For the sake of completeness we finally turn to the calculation of the diagram with the topological structure shown in Fig. 5. This case is less involved than the case of the master diagram, and the result can be expressed by dilogarithms and logarithms. However, some new features appear in connection with the adiabatic limit.

Again, we start from (3.28) and we only consider terms with field operator $:A(2)A(4):$. Additionally, we fix the inner coordinates as shown in Fig. 5. Then the first, the third, the fourth and the fifth term in (3.28) vanish in the adiabatic limit. As we shall see, the second term R'_{42} and the seventh term R'_{47} cancel an infrared divergence occurring in R'_{46} . We directly start with the investigation of the two-particle contribution

$$R'_{42} = r'_{42}(1, 2, 3, 4) :A(2)A(4):, \quad (4.1)$$

$$r'_{42} = 2i\Sigma_0(1-3) D_m^F(3-4) D_m^+(1-2) D_m^+(4-2) :A(2)A(4):, \quad (4.2)$$

$$r'_{42}(p_1, p_2, p_3) = 2i(2\pi)^{-2} \int dp' \Sigma_0(p_1 - p') D_m^F(p_1 + p_3 - p') D_m^+(p') D_m^+(-p_2 - p'), \quad (4.3)$$

where Σ_0 is the electron self-energy in second order for a massless photon, given by [7]

$$\Sigma_0(p) = -\frac{i}{4(2\pi)^4} \frac{p^2 - m^2}{p^2} \log \frac{m^2 - p^2 - i0}{m^2}. \quad (4.4)$$

The theory of distribution splitting leaves us still the possibility to add a constant c_0 to Σ_0 , since Σ_0 has singular order $\omega = 0$. But an investigation of the adiabatic limit of totally scalar QED with a massless photon yields the result that c_0 must vanish, otherwise the limit would not exist (cf. [6]). Moreover, choosing $c_0 \neq 0$ would not change the result for the diagram discussed here, because the added contributions would actually cancel in the end.

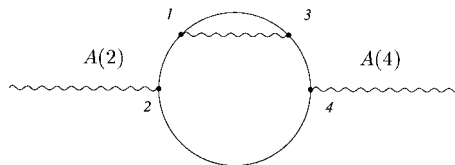


FIG. 5. A simple two-loop.

As mentioned above, there is also a further two-particle cut in R'_{42} with exchanged inner coordinates, giving the same contribution to the final result. Similarly, if the coordinates x_1 and x_3 are exchanged in Fig. 5, then the three-particle cut in R'_{46} vanishes and an analogous diagram occurs in R'_{44} . We will take this fact into consideration by correct normalization of the resulting causal distribution d'_4 . Performing the adiabatic limit in (4.3)

$$r'_{42}(p) = 2i(2\pi)^{-2} \int dp' \Sigma_0(-p') D_m^F(-p') D_m^+(p') D_m^+(-p-p') \quad (4.5)$$

causes a problem, since the singularities of the Feynman propagator and the self-energy are overlapping with the support of one D_m^+ -distribution in a way which makes it difficult to give a meaning to (4.5). Even the introduction of a finite photon mass m_3 in electron self-energy does not help. Therefore, we leave the photon massless and introduce different masses for the electron lines as

$$r'_{42}(p) = 2i(2\pi)^{-2} \int dp' \Sigma_0(-p') D_{m_2}^F(-p') D_{m_1}^+(p') D_m^+(-p-p'). \quad (4.6)$$

Then we arrive at ($p = (p_0, \mathbf{0})$)

$$\begin{aligned} r'_{42}(p) &= \frac{1}{2(2\pi)^{10}} \int dp' \Theta(p'_0) \delta(p'^2 - m_1^2) \Theta(-p_0 - p'_0) \delta((p+p')^2 - m^2) \\ &\quad \times \frac{m_1^2 - m^2}{m_1^2(m_1^2 - m_2^2)} \log \left| \frac{m^2 - m_1^2}{m^2} \right| \\ &= \frac{1}{2(2\pi)^{10}} \Theta(-p_0) \int \frac{d^3p'}{2E} \frac{1}{2|p_0|} \delta(E - |p_0|/2) \\ &\quad \times \frac{m_1^2 - m^2}{m_1^2(m_1^2 - m_2^2)} \log \left| \frac{m^2 - m_1^2}{m^2} \right| + o(m_1^2 - m^2) \\ &= \frac{\delta}{8(2\pi)^9} \Theta(-p_0) \Theta(p^2 - 4m^2) \frac{m_1^2 - m^2}{m_1^2(m_1^2 - m_2^2)} \log \left| \frac{m^2 - m_1^2}{m^2} \right|, \end{aligned} \quad (4.7)$$

$$E = \sqrt{\mathbf{p}'^2 + m^2}, \quad \delta = \sqrt{1 - 4m^2/p^2}. \quad (4.8)$$

The limit $(m_1, m_2) \rightarrow m$ can only be performed if we combine $r'_{42} + r'_{47}$. In order to keep the calculation clearly arranged, we take the special choice for the masses $m_1 = m + \Delta$, $m_2 = m - \Delta$, $\Delta \ll m$. We receive then a very simple result:

$$\begin{aligned}
r'_{42} + r'_{47} &= \frac{\delta}{8(2\pi)^9} \Theta(-p_0) \Theta(p^2 - 4m^2) \\
&\times \left\{ \frac{m_1^2 - m^2}{m_1^2(m_1^2 - m_2^2)} \log \left| \frac{2m\Delta + \Delta^2}{m^2} \right| + \frac{m_2^2 - m^2}{m_2^2(m_2^2 - m_1^2)} \log \left| \frac{2m\Delta - \Delta^2}{m^2} \right| \right\} \\
&= \frac{\delta}{8(2\pi)^9} \Theta(-p_0) \Theta(p^2 - 4m^2) \frac{m^2}{m_1^2 m_2^2} \log \frac{2\Delta}{m} + o(\Delta) \\
&\rightarrow \frac{\Theta(-p_0) \Theta(p^2 - 4m^2)}{8(2\pi)^9} \frac{\delta}{m^2} \log \frac{2\Delta}{m}. \tag{4.9}
\end{aligned}$$

For $\Delta \rightarrow 0$ there is a logarithmic mass singularity.

There remains to calculate the three-particle cut r'_{46} . We directly start from

$$\begin{aligned}
r'_{46}(p) &= \frac{2}{(2\pi)^{11}} \int dp' dq \frac{1}{(p+q)^2 - m_2^2} \frac{1}{(p+q)^2 - m_1^2} \Theta(q_0) \delta(q^2 - m^2) \\
&\times \Theta(p'_0) \delta(p'^2) \Theta(-p-p'-q) \delta((p+p'+q)^2 - m^2), \tag{4.10}
\end{aligned}$$

where the adiabatic limit has already been performed. Since [7, Eq. (3.7.8)]

$$\begin{aligned}
&\int dp' \Theta(p'_0) \delta(p'^2) \Theta(-p_0 - p'_0 - q_0) \delta((p+p'+q)^2 - m^2) \\
&= \frac{\pi}{2} \Theta(-p_0 - q_0) \Theta((p+q)^2 - m^2) \frac{(p+q)^2 - m^2}{(p+q)^2}, \tag{4.11}
\end{aligned}$$

we obtain

$$\begin{aligned}
r'_{46} &= \frac{1}{2(2\pi)^9} \Theta(-p_0) \Theta(p^2 - 4m^2) \int_m^{-p_0/2} dE \sqrt{E^2 - m^2} \\
&\times \frac{p_0^2 + 2p_0 E}{p_0^2 + 2p_0 E + m^2} \frac{1}{(p_0^2 + 2p_0 E + m^2 - m_1^2)(p_0^2 + 2p_0 E + m^2 - m_2^2)}, \tag{4.12}
\end{aligned}$$

for $p = (p_0, \mathbf{0})$ and taking all Θ - and δ -distributions into account. We introduce the integration variable $y = E/|p_0|$ and

$$\Delta_1 = \frac{m^2 - m_1^2}{p_0^2}, \quad \Delta_2 = \frac{m^2 - m_2^2}{p_0^2}, \quad x^2 = m^2/p_0^2. \tag{4.13}$$

Note that the definition of x is not the same as in the calculations for the electron propagator. Then the integral above assumes the form

$$\begin{aligned}
J &= \frac{1}{p_0^2} \int_x^{1/2} dy \sqrt{y^2 - x^2} \left\{ \frac{A_1}{(A_1 - A_2)(x^2 - A_1)} \frac{1}{1 - 2y + A_1} \right. \\
&\quad \left. + \frac{A_2}{(A_2 - A_1)(x^2 - A_2)} \frac{1}{1 - 2y + A_2} + \frac{x^2}{(x^2 - A_1)(x^2 - A_2)} \frac{1}{2y - 1 - x^2} \right\} \\
&= \frac{1}{m^2} \int_x^{1/2} dy \frac{\sqrt{y^2 - x^2}}{1 - 2y - 2x^2 \frac{A}{m}} + \frac{1}{m^2} \int_x^{1/2} dy \frac{\sqrt{y^2 - x^2}}{2y - 1 - x^2} + o(A). \quad (4.14)
\end{aligned}$$

The first integral in (4.14) contains the mass singularity, the second one is obviously finite. The evaluation of (4.14) is best done by means of the substitution

$$y^2 - x^2 = (z' - y)^2, \quad y = \frac{z'^2 + x^2}{2z'}, \quad z' = y + \sqrt{y^2 - x^2}, \quad (4.15)$$

which leads up to terms $o(A)$ to $(z = (3.62))$

$$\begin{aligned}
J &= \frac{\delta}{4m^2} \left[\log(1+z) + 2 \log(1-z) - \frac{3}{2} \log z - \log \frac{2A}{m} \right] + \frac{1 + \delta^2/2}{4m^2} \log z \\
&= \frac{1}{4m^2} \left\{ \frac{1-z}{1+z} \log(1+z) + 2 \frac{1-z}{1+z} \log(1-z) + \left(3 - \frac{5}{z+1} + \frac{2}{(z+1)^2} \right) \log z \right. \\
&\quad \left. + \frac{1-z}{1+z} \log \frac{2A}{m} \right\}. \quad (4.16)
\end{aligned}$$

The splitting of (4.16) can be done the same way as for the master diagram. The splitting of

$$\operatorname{sgn} p_0 \Theta(p^2 - 4m^2) \frac{1-z}{1+z} \log(1 \pm z) \quad (4.17)$$

leads to the integral

$$-\frac{i}{2\pi} R_{\pm} = \frac{i}{2\pi} \int_{2\sqrt{m/p^2}}^{\infty} dt \frac{2t}{1-t^2} \frac{1-x}{1+x} \log(1 \pm x) \quad (4.18)$$

$$= -\frac{i}{2\pi} \int_0^1 dx \left\{ \frac{1}{x} + \frac{z-1}{z+1} \frac{1}{x-z} - \frac{z-1}{z+1} \frac{1}{x-1/z} \right\} \log(1 \pm x) \quad (4.19)$$

after the substitution

$$x = \frac{1 - \sqrt{1 - 4m^2/p^2 t^2}}{1 + \sqrt{1 - 4m^2/p^2 t^2}}.$$

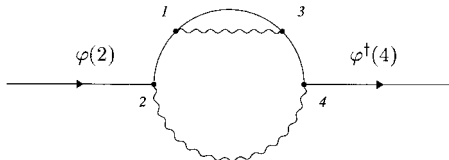


FIG. 6. Another simple two-loop.

Then we have ($z \in [0, 1]$)

$$R_+ = \frac{1-z}{1+z} \left[\text{Li}(z) + \text{Li}(-z) + \log z \log(1+z) - \frac{\pi^2}{12} \right] + \frac{\pi^2}{12}.$$

and

$$R_- = \frac{1-z}{1+z} \left[2\text{Li}(z) + \log z \log(1-z) + \frac{\pi^2}{6} \right] - \frac{\pi^2}{6}.$$

The splitting of

$$\text{sgn } p_0 \Theta(p^2 - 4m^2) \left(3 - \frac{5}{z+1} + \frac{2}{(z+1)^2} \right) \log z \quad (4.20)$$

leads to

$$R_3 = \int_0^1 dx \left\{ \frac{4z}{(z+1)^2} \frac{1}{x+1} + \frac{z+3}{(z+1)^2} \frac{1}{x-1/z} + \frac{z(3z+1)}{(z+1)^2} \frac{1}{x-z} \right\} \log x \quad (4.21)$$

$$= -3 \frac{z-1}{z+1} \text{Li}(z) - \frac{z(3z+1)}{2(z+1)^2} \log^2 z + \frac{z^2 \pi^2}{(z+1)^2}. \quad (4.22)$$

4.2. The Electron Propagator

We will not discuss the calculation of the diagram in Fig. 6 in full detail, since we are in the most simple situation which has occurred so far. Again, we start from (3.28) and we only consider terms with field operator $:\varphi(2)\varphi^\dagger(4):$. We fix the inner coordinates as shown in Fig. 6. The first, the third, the fourth and the fifth term in (3.28) vanish in the adiabatic limit. The second term R'_{42} and the seventh term R'_{47} cancel a mass singularity occurring in R'_{46} . An corresponding statement applies to the diagram where the inner coordinates are exchanged.

Straightforward calculation of

$$r'_{42}(p) = i(2\pi)^{-2} \int dp' \Sigma_0(-p') D_{m_2}^F(-p') D_{m_1}^+(p') D_0^+(-p-p') \quad (4.23)$$

and adding r'_{47} yields, maintaining the conventions $m_1 = m + \Delta$, $m_2 = m - \Delta$,

$$r'_{42} + r'_{47} = \frac{\Theta(-p_0) \Theta(p^2 - m^2) p^2 - m^2}{16(2\pi)^9} \log \frac{2\Delta}{m} + o(\Delta). \quad (4.24)$$

The three-particle cut is given by

$$r'_{46}(p) = \frac{\Theta(-p_0) \Theta(p^2 - m^2)}{16(2\pi)^9 m^2} \left\{ \frac{1 - x^2}{x^2} \log \frac{2\Delta/m}{x^2 - 1} - \log x^2 \right\}, \quad (4.25)$$

$$x^2 = p^2/m^2.$$

The mass singularities cancel, and we can write down the result for the causal distribution which includes all diagrams with external field operators $:\varphi(2) \varphi^\dagger(4):$

$$d'_4(p) = \frac{\text{sgn}(p_0) \Theta(p^2 - m^2)}{8(2\pi)^9 m^2} \left\{ \frac{1 - x^2}{x^2} \log(x^2 - 1) + \log x^2 \right\}. \quad (4.26)$$

Applying the splitting-procedure to (4.26),

$$r(p) = \frac{i}{8(2\pi)^{10} m^2} \int_{-\infty}^{+\infty} dt \frac{\text{sgn } t \Theta(x^2 t^2 - 1)}{1 - t + i0}$$

$$\times \left[\frac{1}{x^2 t^2} \log(x^2 t^2 - 1) + (\log x^2 t^2 - \log(x^2 t^2 - 1)) \right], \quad p \in V^+, \quad (4.27)$$

leads to the result

$$t'_4(p) = \frac{i}{8(2\pi)^{10} m^2} \left[\frac{1}{x^2} \log^2(1 - x^2) - \left(\text{Li}(x^2) + \frac{1}{2} \log^2(1 - x^2) + \frac{\pi^2}{6} \right) \right], \quad (4.28)$$

$$x^2 = \frac{p^2 + i0}{m^2}.$$

APPENDIX A

Free Fields, Commutation Relations, and Propagators

Totally scalar QED:

Charged massive scalar field:

$$\varphi(x) = \varphi^{(-)}(x) + \varphi^{(+)}(x), \quad (\text{A.1})$$

$$\varphi^{(-)}(x) = \frac{1}{(2\pi)^{-3/2}} \int \frac{d^3k}{\sqrt{2k^0}} a(\underline{k}) e^{-ikx}, \quad (\text{A.2})$$

$$\varphi^{(+)}(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2k^0}} b^+(\underline{k}) e^{ikx},$$

$$k^0 = \sqrt{\underline{k}^2 + m^2}. \quad (\text{A.3})$$

Commutation relations:

$$[\varphi(x)^{(\mp)}(x), \varphi^{\dagger(\pm)}(y)] = -iD_m^{\pm}(x-y), \quad (\text{A.4})$$

where

$$\hat{D}_m^{\pm}(p) = \pm \frac{i}{2\pi} \Theta(\pm p^0) \delta(p^2 - m^2). \quad (\text{A.5})$$

Jordan-Pauli distribution:

$$D_m(x) = D_m^+(x) + D_m^-(x). \quad (\text{A.6})$$

Retarded and advanced distributions:

$$D_m^{ret}(x) = \Theta(x^0) D_m(x), \quad D_m^{av}(x) = \Theta(-x^0) D_m(x), \quad (\text{A.7})$$

$$\hat{D}_m^{ret}(p) = -\frac{(2\pi)^{-2}}{p^2 - m^2 + ip^0 0}, \quad \hat{D}_m^{av}(p) = -\frac{(2\pi)^{-2}}{p^2 - m^2 - ip^0 0}. \quad (\text{A.8})$$

Feynman propagator:

$$D_m^F(p) = -\frac{(2\pi)^{-2}}{p^2 - m^2 + i0}. \quad (\text{A.9})$$

Neutral scalar field:

$$A(x) = (2\pi)^{-3/2} \int \frac{d^3k}{\sqrt{2k^0}} [a(\underline{k}) e^{-ikx} + a^{\dagger}(\underline{k}) e^{ikx}], \quad (\text{A.10})$$

$$k^0 = |\underline{k}|, \quad (\text{A.11})$$

$$[A(x), A(y)] = -iD_0(x-y). \quad (\text{A.12})$$

QED:

(Anti-) commutators:

$$\{\psi^{(-)}(x), \bar{\psi}^{(+)}(y)\} = \frac{1}{i} S^{(+)}(x-y) = (i\gamma_{\mu}\partial_x^{\mu} + m) D_m^{(+)}(x-y), \quad (\text{A.13})$$

$$\{\bar{\psi}^{(-)}(x), \psi^{(+)}(y)\} = \frac{1}{i} S^{(-)}(x-y) = (i\gamma_{\mu}\partial_x^{\mu} + m) D_m^{(-)}(x-y), \quad (\text{A.14})$$

$$[A_{\mu}^{(\mp)}(x), A_{\nu}^{(\pm)}(y)] = ig_{\mu\nu} D_0^{(+)}(x-y). \quad (\text{A.15})$$

Propagators:

$$S_m^F(x) = -(i\gamma_{\mu}\partial^{\mu} + m) D_m^F(x). \quad (\text{A.16})$$

QCD:

Commutators for the gauge fields:

$$[A_{\mu}^a(x), A_{\nu}^b(y)] = ig_{\mu\nu}\delta_{ab}D_0(x-y), \quad a, b = 1, \dots, 8. \quad (\text{A.17})$$

Anticommutators for the fermionic ghost fields:

$$\{u_a^{(\mp)}(x), \tilde{u}_b^{(\pm)}(y)\} = -i\delta_{ab}D_0^{(\pm)}(x-y). \quad (\text{A.18})$$

Gravity:

Gravitons:

$$[h_{\alpha\beta}(x), h_{\mu\nu}(y)] = -\frac{i}{2} (g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu} - g^{\alpha\beta}g^{\mu\nu}) D_0(x-y), \quad (\text{A.19})$$

Fermionic vector ghosts:

$$\{u_{\rho}(x), \tilde{u}_{\sigma}(y)\} = ig_{\rho\sigma}D_0(x-y). \quad (\text{A.20})$$

APPENDIX B

Useful Relations for the Calculation of 2-Loop Diagrams

The formula

$$\int^x dt \frac{\log(at+b)}{ct+d} = \frac{1}{c} \log \left| \frac{bc-ad}{c} \right| \log \left| \frac{a(cx+d)}{ad-bc} \right| - \frac{1}{c} \text{Li} \left(\frac{a(cx+d)}{ad-bc} \right),$$

$$a, b, c, d, x \in \mathbf{R}, \quad (\text{B.1})$$

is fully sufficient for the calculation of the causal distribution of all diagrams discussed in the present study except for the three-particle cut which introduces elliptic functions in the electron propagator. Also the integrals appearing in (3.96, 3.97) can be performed by the help of (B.1). The remaining calculations require the following identities, holding for $z \in [0, 1)$,

$$\int_0^z \frac{dy}{y} \operatorname{Li}(y) = \operatorname{Li}_3(z), \quad (\text{B.2})$$

$$\int_0^z \frac{dy}{y} \operatorname{Li}(-y) = \operatorname{Li}_3(-z), \quad (\text{B.3})$$

$$\int_0^z \frac{dy}{y} \log y \log(1-y) = -\operatorname{Li}(z) \log z + \operatorname{Li}_3(z), \quad (\text{B.4})$$

$$\int_0^z \frac{dy}{y} \log y \log(1+y) = -\operatorname{Li}(-z) \log z + \operatorname{Li}_3(-z), \quad (\text{B.5})$$

$$\int_0^z \frac{dy}{y-1} \log^2 y = -2\operatorname{Li}_3(z) + 2\operatorname{Li}(z) \log z + \log(1-z) \log^2 z, \quad (\text{B.6})$$

$$\int_0^z \frac{dy}{y+1} \log^2 y = -2\operatorname{Li}_3(-z) + 2\operatorname{Li}(-z) \log z + \log(1+z) \log^2 z \quad (\text{B.7})$$

in the case of vacuum polarization and

$$\begin{aligned} \int_{1/x^2}^{\infty} \frac{ds}{s(1-s)} \log(sx^2) \log(sx^2-1) &= \int_0^1 \frac{ds}{s-1/x^2} \log s \log\left(\frac{s}{1-s}\right) \\ &= \operatorname{Li}_3(1-x^2) + \frac{1}{2} \log x^2 \log^2(x^2-1) \\ &\quad - \frac{\pi^2}{2} \log x^2 - \zeta(3), \end{aligned} \quad (\text{B.8})$$

$$\begin{aligned} \int_{1/x^2}^{\infty} \frac{ds}{s(1-s)} \log^2(sx^2-1) &= \int_0^1 \frac{ds}{s-1/x^2} \log^2\left(\frac{s}{1-s}\right) \\ &= \frac{1}{3} \log^3(x^2-1) - \frac{2\pi^2}{3} \log(x^2-1), \end{aligned} \quad (\text{B.9})$$

$$\begin{aligned}
\int_{1/x^2}^{\infty} \frac{ds}{s(1-s)} \operatorname{Li}(sx^2) &= \int_0^1 \frac{ds}{s-1/x^2} \left\{ -\operatorname{Li}(s) - \frac{1}{2} \log^2 s + \frac{\pi^2}{3} \right\} \\
&= \operatorname{Li}_3(1-x^2) - \log(x^2-1) \operatorname{Li}(1-x^2) \\
&\quad - \frac{1}{2} \log x^2 \log^2(x^2-1) \\
&\quad + \frac{\pi^2}{2} \log x^2 + \frac{\pi^2}{6} \log(x^2-1) - \zeta(3), \tag{B.10}
\end{aligned}$$

where $x^2 \in [1, \infty)$, for the splitting of the electron propagator.

We emphasize the fact that polylogarithms obey very many identities, so the representation of our expressions containing di- and trilogarithms is by far not unique. It was our goal to get the most compact form of the results.

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