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Power counting degree *vs.* singular order in the Schwinger model

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Summary. — The importance of a rigorous definition of the singular degree of a distribution is demonstrated in the case of two-dimensional QED (Schwinger model). Correct mathematical treatment of second-order vacuum polarization in the perturbative approach is crucial in order to obtain the Schwinger mass of the photon by resummation.

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1. – Introduction

The Schwinger model [1] still serves as a very popular laboratory for quantum field theoretical methods. Although its nonperturbative properties and their relations to confinement [2,3] have always been of greatest interest, it is also possible to discuss the model perturbatively in a straightforward way. The calculation of the vacuum polarization diagram (VP) at second order then turns out to be a delicate task, where a careful discussion of the scaling behaviour of distributions becomes necessary.

We will demonstrate this fact in the framework of causal perturbation theory in the following.

2. – The causal approach

In causal perturbation theory, which goes back to a classical paper by Epstein and Glaser [4], the S-matrix is constructed inductively order by order as an operator-valued functional

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int \mathrm{d}^4x_1 \dots \mathrm{d}^4x_n \, T_n(x_1, \dots x_n) g(x_1) \dots g(x_n), \tag{1}$$

where $g(x)$ is a tempered test function that switches the interaction. The first order (*e.g.* for QED)

$$T_1(x) = ie : \bar{\Psi}(x)\gamma^\mu \Psi(x) : A_\mu(x) \tag{2}$$

must be given in terms of the asymptotic free fields. It is a striking property of the causal approach that *no ultraviolet divergences* appear, *i.e.* the T_n's are finite and well defined. The only remnant of the ordinary renormalization theory is a non-uniqueness of the T_n's due to *finite* normalization terms. The adiabatic limit $g(x) \to 1$ has been shown to exist in purely massive theories at each order [4].

To calculate the second-order distribution T_2, one proceeds as follows: first one constructs the distribution $D_2(x,y)$

$$D_2(x,y) = [T_1(x), T_1(y)] \ , \tag{3}$$

$$\mathrm{supp}\, D_2 = \{(x-y) \mid (x-y)^2 \geq 0\} \ , \tag{4}$$

which has causal support. Then D_2 is split into a retarded and an advanced part $D_2 = R_2 - A_2$, with

$$\mathrm{supp}\, R_2 = \{(x-y) \mid (x-y)^2 \geq 0,\ (x^0 - y^0) \geq 0\} \ , \tag{5}$$

$$\mathrm{supp}\, A_2 = \{(x-y) \mid (x-y)^2 \geq 0,\ -(x^0 - y^0) \geq 0\} \ . \tag{6}$$

Finally T_2 is given by

$$T_2(x,y) = R_2(x,y) + T_1(y)T_1(x) = A_2(x,y) - T_1(x)T_1(y) \ . \tag{7}$$

For the massive Schwinger model with fermion mass m, the part in the Wick-ordered distribution D_2 corresponding to VP

$$D_2(x,y) = e^2[d_2^{\mu\nu}(x-y) - d_2^{\nu\mu}(y-x)] : A_\mu(x)A_\nu(y) : +..., \tag{8}$$

then becomes

$$\hat{d}_2^{\mu\nu}(k) = \frac{1}{2\pi}\int \mathrm{d}^2 z \,, d_2^{\mu\nu}(z)e^{ikz}$$

$$\hat{d}_2^{\mu\nu}(k) = \Big(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\Big)\frac{4m^2}{2\pi}\frac{1}{k^2\sqrt{1-4m^2/k^2}}\mathrm{sgn}(k^0)\Theta(k^2-4m^2) \quad . \tag{9}$$

3. – Power-counting degree and singular order

Obviously, $\hat{d}_2^{\mu\nu}$ has power counting degree $\omega_p = -2$ [6]. But the singular order of the distribution is $\omega = 0$. To show what this means we recall the following definitions [5, 8]:

Definition 1a: The distribution $\hat{d}(p) \in \mathcal{S}'(\mathbf{R}^n)$ has quasi-asymptotics $\hat{d}_0(p) \not\equiv 0$ at $p = \infty$ with respect to *a positive continuous function* $\rho(\delta), \delta > 0$, if the limit

$$\lim_{\delta \to 0} \rho(\delta) \langle \hat{d}\Big(\frac{p}{\delta}\Big), \check{\varphi}(p) \rangle = \langle \hat{d}_0, \check{\varphi} \rangle \tag{10}$$

exists for all $\check{\varphi} \in \mathcal{S}(\mathbf{R}^n)$. The Fourier transform of a test function $\varphi(x)$ is defined by

$$\hat{\varphi}(p) = (2\pi)^{-n/2} \int \mathrm{d}^n x \, \varphi(x) \, e^{ipx} \, . \tag{11}$$

By scaling transformation one derives

$$\lim_{\delta \to 0} \frac{\rho(a\delta)}{\rho(\delta)} = a^{\omega} \equiv \rho_0(\delta) \tag{12}$$

with some real ω. Thus we call $\rho(\delta)$ the power-counting function. The equivalent definition in x-space reads as follows:

Definition 1b: The distribution $d(x) \in \mathcal{S}'(\mathbf{R}^\mathbf{n})$ has *quasi-asymptotics* $d_0(x) \not\equiv 0$ at $x = 0$ with respect to *a positive continuous function* $\rho(\delta)$, $\delta > 0$, if the limit

$$\lim_{\delta \to 0} \rho(\delta) \delta^n d(\delta x) = d_0(x) \tag{13}$$

exists in $\mathcal{S}'(\mathbf{R}^\mathbf{n})$.

Definition 2: The distribution $d(x) \in \mathcal{S}'(\mathbf{R}^n)$ is called *singular of order* ω, if it has quasi-asymptotics $d_0(x)$ at $x = 0$, or its Fourier transform has quasi-asymptotics $\hat{d}_0(p)$ at $p = \infty$, respectively, with power-counting function $\rho(\delta)$ satisfying

$$\lim_{\delta \to 0} \frac{\rho(a\delta)}{\rho(\delta)} = a^{\omega} \, , \quad \forall a > 0. \tag{14}$$

Equation (12) implies

$$\begin{aligned} a^n \langle \hat{d}_0(p), \check{\varphi}(ap) \rangle &= \langle \hat{d}_0\Big(\frac{p}{a}\Big), \check{\varphi}(p) \rangle = a^{-\omega} \langle \hat{d}_0(p), \check{\varphi}(p) \rangle \\ &= \langle d_0(x), \varphi\Big(\frac{x}{a}\Big) \rangle = a^n \langle d_0(ax), \varphi(x) \rangle = a^{-\omega} \langle d_0(x), \varphi(x) \rangle \, , \end{aligned}$$

i.e. $\hat{d}_0$ is homogeneous of degree ω:

$$\hat{d}_0\Big(\frac{p}{a}\Big) = a^{-\omega} \hat{d}_0(p), \tag{15}$$

$$d_0(ax) = a^{-(n+\omega)} d_0(x). \tag{16}$$

This implies that d_0 has power-counting function $\rho(\delta) = \delta^\omega$ and the singular order ω, too. In particular, we have the following estimates for $\rho(\delta)$ [5]: If $\epsilon > 0$ is an arbitrarily small number, then there exist constants C, C' and δ_0 such that

$$C\delta^{\omega+\epsilon} \geq \rho(\delta) \geq C'\delta^{\omega-\epsilon} \quad , \quad \delta < \delta_0. \tag{17}$$

Applying the above definitions to $\hat{d}_2^{\mu\nu}(k)$, we obtain after a short calculation the quasi-asymptotics

$$\lim_{\delta \to 0} \hat{d}_2^{\mu\nu}(k/\delta) = \frac{1}{2\pi}\Big(g^{\mu\nu}k^2 - k^\mu k^\nu\Big)\delta(k^2)\mathrm{sgn}(k^0) \ , \tag{18}$$

and we have $\rho(\delta) = 1$, hence $\omega = 0$. Note that the $g^{\mu\nu}$-term in (18) does not contribute to the quasi-asymptotics. The reason for the result (18) can be explained by the existence of a *sum rule* [7]

$$\int\limits_{4m^2\delta^2}^{\infty} d(q^2) \frac{\delta^2 m^2}{q^4\sqrt{1 - \frac{4m^2\delta^2}{q^2}}} = \frac{1}{2} \ , \tag{19}$$

so that the l.h.s. of (18) is weakly convergent to the r.h.s. In spite of $\mathrm{sgn}(k^0)$, the r.h.s. of (18) is a well-defined tempered distribution due to the factor $(g^{\mu\nu}k^2 - k^\mu k^\nu)$.

This has the following consequence: the retarded part $r_2^{\mu\nu}$ of $d_2^{\mu\nu}$ would be given in the case $\omega < 0$ by the unsubtracted splitting formula

$$\begin{aligned} r_2^{\mu\nu}(k) &= \frac{i}{2\pi}\int\limits_{-\infty}^{\infty} \frac{\mathrm{d}t}{1 - t + i0} d_2^{\mu\nu}(tk) \\ &= \frac{im^2}{\pi^2}\Big(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\Big)\frac{1}{k^2\sqrt{1-4m^2/k^2}} \log\frac{\sqrt{1-4m^2/k^2}+1}{\sqrt{1-4m^2/k^2}-1} \ , \\ &\quad k^2 > 4m^2, k^0 > 0. \end{aligned} \tag{20}$$

This distribution will vanish in the limit $m \to 0$, and the photon would remain massless. But since we have $\omega = 0$, the subtracted splitting formula [5] *must* be used:

$$\begin{aligned} r_2^{\mu\nu}(k) &= \frac{i}{2\pi}\int\limits_{-\infty}^{\infty} \frac{\mathrm{d}t}{(t-i0)^{\omega+1}(1 - t + i0)} d_2^{\mu\nu}(tk) \\ &= \frac{im^2}{\pi^2}\Big(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\Big)\Big(\frac{1}{k^2\sqrt{1-4m^2/k^2}} \log\frac{\sqrt{1-4m^2/k^2}+1}{\sqrt{1-4m^2/k^2}-1} + \frac{1}{2m^2}\Big) \ , \\ &\quad k^2 > 4m^2, k^0 > 0 \ . \end{aligned} \tag{21}$$

The new last term survives in the limit $m \to 0$. After resummation of the VP bubbles it gives the well-known Schwinger mass $m_s^2 = e^2/\pi$ of the photon. Consequently, the difference between simple power-counting and the correct determination of the singular order is by no means a mathematical detail, it is terribly important for the physics.

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