Power counting degree vs. singular order in the Schwinger model

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Summary. — The importance of a rigorous definition of the singular degree of a distribution is demonstrated in the case of two-dimensional QED (Schwinger model). Correct mathematical treatment of second-order vacuum polarization in the perturbative approach is crucial in order to obtain the Schwinger mass of the photon by resummation.

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1. – Introduction

The Schwinger model [1] still serves as a very popular laboratory for quantum field theoretical methods. Although its nonperturbative properties and their relations to confinement [2, 3] have always been of greatest interest, it is also possible to discuss the model perturbatively in a straightforward way. The calculation of the vacuum polarization diagram (VP) at second order then turns out to be a delicate task, where a careful discussion of the scaling behaviour of distributions becomes necessary.

We will demonstrate this fact in the framework of causal perturbation theory in the following.

2. – The causal approach

In causal perturbation theory, which goes back to a classical paper by Epstein and Glaser [4], the $S$-matrix is constructed inductively order by order as an operator-valued functional

$$S(g) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n \, T_n(x_1, \ldots x_n) g(x_1) \ldots g(x_n),$$

(1)
where $g(x)$ is a tempered test function that switches the interaction. The first order (e.g. for QED)\[ T_1(x) = ie : \bar{\Psi}(x)\gamma^\mu \Psi(x) : A_{\mu}(x) \]
must be given in terms of the asymptotic free fields. It is a striking property of the causal approach that no ultraviolet divergences appear, i.e. the $T_n$'s are finite and well defined. The only remnant of the ordinary renormalization theory is a non-uniqueness of the $T_n$'s due to finite normalization terms. The adiabatic limit $g(x) \to 1$ has been shown to exist in purely massive theories at each order \[4\].

To calculate the second-order distribution $T_2$, one proceeds as follows: first one constructs the distribution $D_2(x, y)$\[ D_2(x, y) = [T_1(x), T_1(y)] , \]
which has causal support. Then $D_2$ is split into a retarded and an advanced part $D_2 = R_2 - A_2$, with\[ D_2(x, y) = e^{2[id_2^{\mu\nu}(x - y) - id_2^{\mu\nu}(y - x)]: A_{\mu}(x)A_{\nu}(y) : + ...} , \]
then becomes\[ R_2^{\mu\nu}(k) = \frac{1}{2\pi} \int d^2 z : d_2^{\mu\nu}(z)e^{ikz} \]
\[ A_2^{\mu\nu}(k) = \left( g_{\mu\nu} - \frac{k_{\mu}k_{\nu}}{k^2} \right) \frac{4m^2}{2\pi} \frac{1}{k^2 \sqrt{1 - 4m^2/k^2}} \text{sgn}(k^0) \Theta(k^2 - 4m^2) . \]
3. - Power-counting degree and singular order

Obviously, $\hat{d}_n^{\mu\nu}$ has power counting degree $\omega_p = -2$ [6]. But the singular order of the distribution is $\omega = 0$. To show what this means we recall the following definitions [5, 8]:

Definition 1a: The distribution $\hat{d}(p) \in S'(\mathbb{R}^n)$ has quasi-asymptotics $\hat{d}_0(p) \neq 0$ at $p = \infty$ with respect to a positive continuous function $\rho(\delta), \delta > 0,$ if the limit

$$\lim_{\delta \to 0} \rho(\delta) \langle d\left(\frac{p}{\delta}\right), \varphi(p) \rangle = \langle \hat{d}_0, \varphi \rangle$$

exists for all $\varphi \in S(\mathbb{R}^n)$. The Fourier transform of a test function $\varphi(x)$ is defined by

$$\hat{\varphi}(p) = (2\pi)^{-n/2} \int d^n x \varphi(x) e^{ipx}.$$  

By scaling transformation one derives

$$\lim_{\delta \to 0} \rho(\delta) \frac{a(\delta)}{\rho(\delta)} = a^\omega \equiv \rho_0(\delta)$$

with some real $\omega$. Thus we call $\rho(\delta)$ the power-counting function. The equivalent definition in $x$-space reads as follows:

Definition 1b: The distribution $d(x) \in S'(\mathbb{R}^n)$ has quasi-asymptotics $d_0(x) \neq 0$ at $x = 0$ with respect to a positive continuous function $\rho(\delta), \delta > 0,$ if the limit

$$\lim_{\delta \to 0} \rho(\delta) \delta^n d(\delta x) = d_0(x)$$

exists in $S'(\mathbb{R}^n)$.

Definition 2: The distribution $d(x) \in S'(\mathbb{R}^n)$ is called singular of order $\omega$, if it has quasi-asymptotics $d_0(x)$ at $x = 0$, or its Fourier transform has quasi-asymptotics $d_0(p)$ at $p = \infty$, respectively, with power-counting function $\rho(\delta)$ satisfying

$$\lim_{\delta \to 0} \frac{\rho(a\delta)}{\rho(\delta)} = a^\omega, \quad \forall a > 0.$$ 

Equation (12) implies

$$a^n \langle \hat{d}_0(p), \varphi(ap) \rangle = \langle \hat{d}_0\left(\frac{p}{a}\right), \varphi(p) \rangle = a^{-\omega} \langle \hat{d}_0(p), \varphi(p) \rangle$$

$$= \langle d_0(x), \varphi\left(\frac{x}{a}\right) \rangle = a^n \langle d_0(ax), \varphi(x) \rangle = a^{-\omega} \langle d_0(x), \varphi(x) \rangle,$$

i.e. $\hat{d}_0$ is homogeneous of degree $\omega$:

$$\hat{d}_0\left(\frac{p}{a}\right) = a^{-\omega} \hat{d}_0(p).$$

(15)
order is by no means a mathematical detail, it is terribly important for the physics. The difference between simple power-counting and the correct determination of the singular terms survives in the limit.

\[ C \delta^{\omega+\epsilon} \geq \rho(\delta) \geq C' \delta^{\omega-\epsilon}, \quad \delta < \delta_0. \]

Applying the above definitions to \( d_2^{\mu\nu}(k) \), we obtain after a short calculation the quasi-asymptotics

\[ \lim_{\delta \to 0} d_2^{\mu\nu}(k/\delta) = \frac{1}{2\pi} \left( g^{\mu\nu} k^2 - k^\mu k^\nu \right) \delta(k^2) \text{sgn}(k^0), \]

and we have \( \rho(\delta) = 1 \), hence \( \omega = 0 \). Note that the \( g^{\mu\nu} \) term in (18) does not contribute to the quasi-asymptotics. The reason for the result (18) can be explained by the existence of a sum rule [7]

\[ \int_{4m^2}^{\infty} \frac{d(q^2)}{q^4} \frac{\delta^2 m^2}{\sqrt{1 - 4m^2/q^2}} = \frac{1}{2}, \]

so that the l.h.s. of (18) is weakly convergent to the r.h.s. In spite of \( \text{sgn}(k^0) \), the r.h.s. of (18) is a well-defined tempered distribution due to the factor \( (g^{\mu\nu} k^2 - k^\mu k^\nu) \).

This has the following consequence: the retarded part \( r_2^{\mu\nu}(k) \) of \( d_2^{\mu\nu} \) would be given in the case \( \omega < 0 \) by the unsubtracted splitting formula

\[ r_2^{\mu\nu}(k) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{1 - t + i0} d_2^{\mu\nu}(tk) \]

\[ = \frac{i m^2}{\pi^2} \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \frac{1}{k^2 \sqrt{1 - 4m^2/k^2}} \log \frac{\sqrt{1 - 4m^2/k^2} + 1}{\sqrt{1 - 4m^2/k^2} - 1}, \]

\[ k^2 > 4m^2, k^0 > 0. \]

This distribution will vanish in the limit \( m \to 0 \), and the photon would remain massless. But since we have \( \omega = 0 \), the subtracted splitting formula [5] must be used:

\[ r_2^{\mu\nu}(k) = \frac{i}{2\pi} \int_{-\infty}^{\infty} \frac{dt}{(t - i0)^{\omega+1}} \left( 1 - t + i0 \right) d_2^{\mu\nu}(tk) \]

\[ = \frac{i m^2}{\pi^2} \left( g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \frac{1}{k^2 \sqrt{1 - 4m^2/k^2}} \log \frac{\sqrt{1 - 4m^2/k^2} + 1}{\sqrt{1 - 4m^2/k^2} - 1} + \frac{1}{2m^2}, \]

\[ k^2 > 4m^2, k^0 > 0. \]

The new last term survives in the limit \( m \to 0 \). After resummation of the VP bubbles it gives the well-known Schwinger mass \( m_s^2 = e^2/\pi \) of the photon. Consequently, the difference between simple power-counting and the correct determination of the singular order is by no means a mathematical detail, it is terribly important for the physics.
REFERENCES