Exercises: Mathematical Methods oin Particle Physics

Series 3: *The Classical Scalar Field and the Klein-Gordon Equation*

1. Scalar Model Theory (SM) of Electrodynamics: If one couples a massive *charged* scalar field φ to a *massless* and *neutral* scalar field A via a simplest possible real interaction term $\mathscr{L}_{int}$ according to the Lagrangian density ($\dagger \sim$ complex conjugation)

$$\mathscr{L}_{SM} = \partial_\mu \varphi^\dagger \partial^\mu \varphi - m^2 \varphi^\dagger \varphi + \frac{1}{2}\partial_\mu A \partial^\mu A + \underbrace{g \varphi^\dagger \varphi A}_{\mathscr{L}_{int}},$$

one defines a scalar theory that resembles, in a certain sense, the (quantum) electrodynamics of a charged massive (anti-)particle (e.g., the electron and the positron) interacting with the electromagnetic radiation field associated with massless photons. Here, g is the (dimensional) coupling constant.

Provide the equations of motion for φ, $\varphi^\dagger$, and A that follow from the Euler-Lagrange equations. Does the Klein-Gordon current density $j^\mu = ig(\varphi^\dagger \partial^\mu \varphi - \varphi \partial^\mu \varphi^\dagger)$ satisfy the continuity equation $\partial_\mu j^\mu = 0$?

2. The positive-definite energy density of the real (neutral) free Klein-Gordon field is given, with the standard normalization used in the literature, by

$$T_n^{00} = \frac{1}{2}\dot{\varphi}_n^2 + \frac{1}{2}(\vec{\nabla}\varphi_n)^2 + \frac{m^2}{2}\varphi_n^2 .$$

Let us consider a real plane wave in a cube $W = \{0 \le x^k \le L \mid k = 1,2,3\}$ with volume $V = L^3$ and periodic boundary conditions $\varphi_n(x^0, 0, x^2, x^3) = \varphi_n(x^0, L, x^2, x^3)$, $\varphi_n(x^0, x^1, 0, x^3) = \varphi_n(x^0, x^1, L, x^3)$, and $\varphi_n(x^0, x^1, x^2, 0) = \varphi_n(x^0, x^1, x^2, L)$. Then, with a normalization factor N_n to be determined, a single-particle wave function can be represented by ($kx = k_\mu x^\mu$)

$$\varphi_n(x) = N_n\big(e^{-ikx} + e^{+ikx}\big) = 2N_n \cos kx$$

with suitable wavenumbers $\vec{k} = 2\pi\vec{n}/L$, $\vec{n} \in \mathbb{Z}^3$, and the particle energy $k^0 = \sqrt{\vec{k}^2 + m^2}$.

Calculate the normalization factor N_n such that the energy of the field within the volume of the cube corresponds to the single-particle energy $k^0 = \sqrt{\vec{k}^2 + m^2}$.

3. Due to the absence of a mass term $\sim m^2\varphi^2$ defining a length scale, the massless φ^4 theory, as a classical field theory, exhibits a symmetry under a scale transformation ($\mathbb{R} \ni \alpha \neq 0$)

$$\varphi(x) \mapsto \varphi'(x) = \sigma(\alpha)\varphi(\alpha x) ,$$

in the sense that the action $S_{\varphi_0^4} = \int d^4x \, \mathscr{L}_{\varphi_0^4}(x)$ with the Lagrangian density

$$\mathscr{L}_{\varphi_0^4}(\varphi, \partial_\mu\varphi) = \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{\lambda}{4!}\varphi^4$$

is invariant under the scale transformation.

Determine $\sigma(\alpha)$ and show that the (classical) current linked to the scale symmetry via Noether's theorem

$$j^\mu = \partial^\mu\varphi(x^\nu\partial_\nu\varphi + \varphi) - \frac{1}{2}x^\mu\partial_\nu\varphi\partial^\nu\varphi + \frac{\lambda}{4!}x^\mu\varphi^4$$

is conserved.

Submission of the exercises in mid-November 2026, or by arrangement.