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NON-ABELIAN GAUGE THEORIES AS A CONSEQUENCE OF PERTURBATIVE QUANTUM GAUGE INVARIANCE*

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We show for the case of interacting massless vector bosons, how the structure of Yang–Mills theories emerges automatically from a more fundamental concept, namely perturbative quantum gauge invariance. It turns out that the coupling in a non-Abelian gauge theory is necessarily of Yang–Mills type plus divergence- and coboundary-couplings. The extension of the method to massive gauge theories is briefly discussed.

Keywords: Field theory, S-matrix theory, General properties of quantum chromodynamics.

1. Introduction

It is a well-known fact that gauge theories have many pleasing features. Apart from the beauty of the mathematical structure of the classical theories, it is the renormalizability of their quantum versions which makes them so important for particle physicists. What is less known is that some basic properties of non-Abelian quantum gauge theories can be understood as the consequence of a suitable quantum version of gauge invariance. It is the aim of this paper to demonstrate this fact for the case of interacting massless vector bosons. The result will be the usual Yang–Mills plus ghost couplings plus some divergence- and coboundary-couplings. The nice feature of this approach is that the Lie algebra structure of the Yang–Mills theory is not put in, but comes out: the antisymmetry of the structure constants follows from first order gauge invariance and the Jacobi identity from second order. This fact has been noticed by R. Stora in a more special setting.[1] The present method has already been used by us to analyze the Abelian Higgs model[2] and the electroweak theory.[3,4]

The paper is organized as follows: In Sec. 2 we present our general framework and define perturbative quantum gauge invariance on the Fock space of free asymptotic massless vector bosons and scalar fermionic ghosts.[5,6] In Sec. 3 we investigate how

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gauge invariance restricts all renormalizable couplings of the gauge fields at first and second order of perturbation theory. In Sec. 4 we give a short outlook how the presented method can be generalized to the case of massive gauge bosons.

2. Theoretical Framework

2.1. *Construction of the S matrix*

In this paper, we consider the perturbative S matrix as a sum of smeared operator-valued distributions of the following form:[7,8]

$$S(g) = \mathbf{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int dx_1 \cdots dx_n T_n(x_1, \dots, x_n) g(x_1) \cdot \cdots g(x_n)\,, \tag{1}$$

where $g \in \mathcal{S}$, the Schwartz space of functions of rapid decrease. The test function g plays the role of "adiabatic switching" and provides a cutoff in the long-range part of the interaction, without destroying any symmetry. It can be considered as a natural infrared regulator. We work with free asymptotic fields throughout, so that all expressions are well defined; interacting fields nowhere appear.

The $T_n(x_1, \dots, x_n)$ are well-defined time-ordered products of the first order coupling $T_1(x)$, they are expressed in terms of Wick monomials of free fields. The construction of the T_n requires some care: If the arguments $(x_1, \dots, x_n)$ are all time-ordered, i.e. if we have

$$x_1^0 > x_2^0 > \cdots > x_n^0\,, \tag{2}$$

then T_n is simply given by

$$T_n(x_1, \dots, x_n) = T_1(x_1) T_1(x_2) \cdot \cdots T_1(x_n)\,. \tag{3}$$

According to the definition (1) $T_n(x_1, \dots, x_n)$ is symmetric in $x_1, \dots, x_n$. Using this fact allows us in principle to obtain the operator-valued distribution T_n inductively everywhere except for the complete diagonal $\Delta_n = \{x_1 = \cdots = x_n\}$.[9] If T_n were a C number distribution, we could make it a well-defined distribution for all $x_1, \dots, x_n$ by extending the distribution from R^{4n}/Δ_n to R^4. The problem can be reduced to a C number problem by the Wick expansion of the operator-valued distributions. The extension $T(x_1, \dots, x_n)$ is, of course, not unique: it is ambiguous up to distributions with local support Δ_n. This ambiguity can be further reduced by the help of symmetries (in particular gauge invariance) and power counting theory.

The concrete inductive construction of the T_n has also to be performed carefully. Since the behavior of the distributions in p-space is much better than the very singular one in x-space, it is advantageous to use the original Epstein–Glaser method of splitting causal distributions,[8,10] instead of the above extension method, because the former can be translated to p-space. The well-known ultraviolet divergences are in fact a consequence of "careless" splitting of operator valued distributions.

2.2. *Definition of perturbative quantum gauge invariance*

We introduce the concept of gauge invariance for the simple case of quantum electrodynamics first, where the coupling of the electron to the photon is given at first order by

$$T_1(x) = ie : \bar{\Psi}(x)\gamma^\mu \Psi(x) : A_\mu(x) \,. \tag{4}$$

Let

$$Q \stackrel{\text{def}}{=} \int d^3x \big(\partial_\mu A^\mu(x) \overset{\leftrightarrow}{\partial_0} u(x)\big) \tag{5}$$

be the generator of (free) gauge transformations, called gauge charge for brevity. This Q has first been introduced in a paper by T. Kugo and I. Ojima.[11] A_μ is the gauge potential in the Feynman gauge, and we choose u as a scalar fermionic field, in order to have $Q^2 = 0$ (see Subsec. 2.3). The free fields satisfy the well-known commutation relations

$$\big[A_\mu^{(\pm)}(x), A_\nu^{(\mp)}(y)\big] = ig^{\mu\nu} D^{(\mp)}(x-y) \,, \tag{6}$$

$$\big\{u^{(\pm)}(x), \tilde{u}^{(\mp)}(y)\big\} = -iD^{(\mp)}(x-y) \,, \tag{7}$$

and all other commutators vanish. $D^\mp$ are the well-known positive and negative frequency parts of the Pauli–Jordan distribution. All the fields fulfill the Klein–Gordon equation with zero mass, and as already mentioned, we are working in Feynman gauge, but we are not forced to do so. The following discussion would go through with some technical changes in other covariant ξ-gauges as well.[12] In order to see how the infinitesimal gauge transformation acts on the free fields, we calculate the commutators[5]

$$\begin{aligned} [Q, A_\mu] &= i\partial_\mu u \,, & \{Q, u\} &= 0 \,, \\ \{Q, \tilde{u}\} &= -i\partial_\nu A^\nu \,, & [Q, \Psi] &= [Q, \bar{\Psi}] = 0 \,. \end{aligned} \tag{8}$$

Then we have

$$[Q, T_1(x)] = -e : \bar{\Psi}\gamma^\mu\Psi : \partial_\mu u = i\partial_\mu(ie : \bar{\Psi}\gamma^\mu\Psi : u) = i\partial_\mu T^\mu_{1/1}(x) \,. \tag{9}$$

Note that the free electron field is *not* affected by the gauge transformation. We will call $T^\mu_{1/1}$ the "Q-vertex" in the sequel. The generalization of (9) to nth order is

$$[Q, T_n(x_1, \ldots, x_n)] = i\sum_{l=1}^{n} \partial_\mu^{x_l} T^\mu_{n/l}(x_1, \ldots, x_n) = (\text{sum of divergences}) \,, \tag{10}$$

where $T^\mu_{n/l}$ is a mathematically rigorous version of the time-ordered product

$$T^\mu_{n/l}(x_1, \ldots, x_n) \text{ “=” } T(T_1(x_1) \cdots T^\mu_{1/1}(x_l) \cdots T_1(x_n)) \,, \tag{11}$$

constructed by means of the methods described in Subsec. 2.1. We define (10) to be the condition of gauge invariance.[5] For a fixed x_l we consider from T_n all terms with the external field operator $A_\mu(x_l)$

$$T_n(x_1, \ldots, x_n) = : t_l^\mu(x_1, \ldots, x_n) A_\mu(x_l) : + \cdots \tag{12}$$

(the dots represent terms without $A_\mu(x_l)$). Then gauge invariance (10) requires

$$\partial_\mu^l[t_l^\mu(x_1,\dots,x_n)u(x_l)] = t_l^\mu(x_1,\dots,x_n)\partial_\mu u(x_l) \tag{13}$$

or

$$\partial_\mu^l t_l^\mu(x_1,\dots,x_n) = 0\,, \tag{14}$$

i.e. we obtain the Ward–Takahashi identities[13] for QED.

As a further step, we consider the Yang–Mills theory without fermions now. The first order coupling shall be given by

$$T_1(x) = igf_{abc}\left\{\frac{1}{2}:A_{\mu a}(x)A_{\nu b}(x)F_c^{\nu\mu}(x): - :A_{\mu a}(x)u_b(x)\partial^\mu\tilde{u}_c(x):\right\}, \tag{15}$$

where $F_a^{\nu\mu} = \partial^\nu A_a^\nu - \partial^\mu A_a^\nu$ is the free field strength tensor, u_a, $\tilde{u}_a$ are the (fermionic) ghost fields, and the f_{abc} are the usual SU(N) structure constants. The asymptotic free fields satisfy the commutation relations

$$\left[A_{\mu a}^{(\pm)}(x), A_{\nu b}^{(\mp)}(y)\right] = i\delta_{ab}g_{\mu\nu}D^{(\mp)}(x-y) \tag{16}$$

and

$$\left\{u_a^{(\pm)}(x), \tilde{u}_b^{(\mp)}(y)\right\} = -i\delta_{ab}D^{(\mp)}(x-y)\,, \tag{17}$$

and all other {anti-}commutators vanish. The introduction of ghost couplings is necessary here to preserve perturbative quantum gauge invariance at first order.

Defining the gauge charge as in (5) by

$$Q := \int d^3x\,\partial_\mu A_a^\mu(x)\overset{\leftrightarrow}{\partial_0} u_a(x)\,, \tag{18}$$

where summation over repeated indices is understood, we are led to the following (anti-)commutators with the fields:

$$\begin{aligned} &[Q, A_a^\mu] = i\partial^\mu u_a\,, && [Q, F_a^{\mu\nu}] = 0\,,\\ &\{Q, u_a\} = 0\,, && \{Q, \tilde{u}_a\} = -i\partial_\mu A_a^\mu\,. \end{aligned} \tag{19}$$

Obviously, these gauge variations have a "simpler" structure than the well-known BRS transformation, because the non-Abelian parts are missing here. Calculating $[Q, T_1(x)]$ gives the Q-vertex

$$T_{1/1}^\mu = igf_{abc}\left\{:u_a A_{\nu b}F_c^{\mu\nu}: + \frac{1}{2}:u_a u_b\partial^\mu\tilde{u}_c:\right\} \tag{20}$$

and the first order coupling is gauge invariant through the presence of a ghost coupling term in T_1. The first order coupling given in (15) is in fact not the most general one, and it is also possible to construct a gauge-invariant first order coupling with bosonic ghosts;[5] but gauge invariance then breaks down at second order of perturbation theory. Gauge invariance at order n is again defined by (11).

Note that the usual four-gluon term $\sim g^2$ is missing in T_1. This term appears as a necessary local normalization term at second order, as we shall see (67). This stresses the fact that our notion of perturbative gauge invariance is strongly related to the formal expansion of the theory in powers of the coupling constant g, in contrast to the conventional treatment which discusses quantum effects of a local gauge group via formal expansions in powers of $\hbar$. In our approach, no local gauge group is assumed from the start. Local gauge groups will rather appear as a consequence of perturbative gauge invariance.

2.3. *Important properties of the gauge charge Q*

Using the Leibnitz rule for graded algebras gives

$$\begin{aligned} Q^2 &= \frac{1}{2}\{Q,Q\} \\ &= \frac{1}{2}\int_{x_0=\mathrm{const}} d^3x\, \partial_\nu A_a^\nu(x)\{\overleftrightarrow{\partial}_{x_0} u_a(x), Q\} \\ &\quad -\frac{1}{2}\int_{x_0=\mathrm{const}} d^3x[\partial_\nu A_a^\nu(x), Q]\,\overleftrightarrow{\partial}_{x_0} u_a(x) = 0\,, \end{aligned} \tag{21}$$

i.e. Q is nilpotent. This basic property of Q and the Krein structure on the Fock–Hilbert space[14,15] allows us to prove unitarity of the S matrix on the physical Hilbert space H_{phys}, which is a subspace of the Fock–Hilbert space F containing also the unphysical ghosts and unphysical degrees of freedom of the vector field.[6]

The physical Fock space can be expressed by the kernel and the range of Q,[6,14] namely

$$H_{\mathrm{phys}} = \ker Q/\mathrm{ran}\, Q = \ker\{Q, Q^+\}\,. \tag{22}$$

This can be most easily seen by realizing the various field operators on a *positive definite* Fock–Hilbert space F as follows:

$$A_0(x) = (2\pi)^{-3/2}\int \frac{d^3p}{\sqrt{2p^0}}\big[a_0(\mathbf{p})e^{-ipx} - a_0^+(\mathbf{p})e^{ipx}\big]\,, \tag{23}$$

$$A_j(x) = (2\pi)^{-3/2}\int \frac{d^3p}{\sqrt{2p^0}}\big[a_j(\mathbf{p})e^{-ipx} + a_j^+(\mathbf{p})e^{ipx}\big] \quad \text{for} \quad j=1,2,3\,, \tag{24}$$

$$u(x) = (2\pi)^{-3/2}\int \frac{d^3p}{\sqrt{2p^0}}\big(c_2(\mathbf{p})e^{-ipx} + c_1^+(\mathbf{p})e^{ipx}\big)\,, \tag{25}$$

$$\tilde{u}(x) = (2\pi)^{-3/2}\int \frac{d^3p}{\sqrt{2p^0}}\big(-c_1(\mathbf{p})e^{-ipx} + c_2^+(\mathbf{p})e^{ipx}\big)\,, \tag{26}$$

and

$$[a_\nu(\mathbf{p}), a_\mu^+(\mathbf{p})] = \delta_{\mu\nu}\delta(\mathbf{p}-\mathbf{q})\,, \qquad \mu = 0,1,2,3\,, \tag{27}$$

$$\{c_i(\mathbf{p}), c_j^+(\mathbf{q})\} = \delta_{ij}\delta(\mathbf{p}-\mathbf{q})\,, \quad i,j=1,2\,. \tag{28}$$

Above, $^+$ denotes the adjoint with respect to the positive scalar product so that the operators can be represented in the usual way in the Fock space. The Krein structure is then defined by introducing the conjugation K

$$a_0(\mathbf{p})^K = -a_0(\mathbf{p})^+ , \qquad a_j(\mathbf{p})^K = a_j(\mathbf{p})^+ , \tag{29}$$

so that $A_\mu^K = A_\mu$, and on the ghost sector

$$c_2(\mathbf{p})^K = c_1(\mathbf{p})^+ , \qquad c_1(\mathbf{p})^K = c_2(\mathbf{p})^+ , \tag{30}$$

so that $u^K = u$ is K-self-adjoint and $\tilde{u}^K = -\tilde{u}$. Then Q is densely defined and becomes K-symmetric $Q \subset Q^K$.

The anticommutator in (22) is essentially the number operator for unphysical particles

$$\{Q^+, Q\} = 2\int d^3p\, \mathbf{p}^2 \left[b_1^+(\mathbf{p})b_1(\mathbf{p}) + b_2^+(\mathbf{p})b_2(\mathbf{p}) + c_1^+(\mathbf{p})c_1(\mathbf{p}) + c_2^+(\mathbf{p})c_2(\mathbf{p})\right] , \tag{31}$$

with

$$b_{1,2} = \frac{a_\parallel \pm a_0}{\sqrt{2}} , \qquad a_\parallel = \frac{p_j a_j}{|\mathbf{p}|} , \tag{32}$$

which implies (22).

The nilpotency of Q allows for standard homological notions:[6] Consider the field algebra $\mathcal{F}$ consisting of the polynomials in the (smeared) gauge and ghost fields and their Wick powers. Defining a gauge variation for a Wick monomial F according to

$$d_Q F \stackrel{\text{def}}{=} QF - (-1)^{n_F} FQ , \tag{33}$$

where n_F is the number of ghost fields in F, Q becomes a differential operator in the sense of homological algebra, and we have

$$d_Q^2 = 0 \Leftrightarrow \{Q, [Q, F_b]\} = [Q, \{Q, F_f\}] = 0 , \tag{34}$$

where F_b is a bosonic and F_f a fermionic operator and $d_Q(FG) = (d_Q F)G + (-1)^{n_F} F d_Q G$. For example, we get

$$d_Q :A_{\mu a} u_b \partial^\mu \tilde{u}_c: = :[Q, A_{\mu a}] u_b \partial^\mu \tilde{u}_c: + :A_{\mu a}[Q, u_b]\partial^\mu \tilde{u}_c: - :A_{\mu a} u_b \{Q, \partial^\mu \tilde{u}_c\}: . \tag{35}$$

If $F = d_Q G$, then F is called a coboundary.

3. Implications of Perturbative Quantum Gauge Invariance

3.1. *First order gauge invariance*

A theory with massless fields usually suffers from serious infrared divergences which may have deep relevance to the confinement mechanism. This, however, is of no importance in our context, since perturbative gauge invariance concerns the T_n only.

We describe the interaction of the massless "gluons" by the most general renormalizable ansatz with zero ghost number (coupling with nonzero ghost number would affect the theory only for unphysical processes)

$$\tilde{T}_1(x) = ig\big\{\tilde{f}^1_{abc} :A_{\mu a}(x)A_{\nu b}(x)\partial^\nu A^\mu_c(x): + \tilde{f}^2_{abc} :A_{\mu a}u_b\partial^\mu\tilde{u}_c: + \tilde{f}^3_{abc} :A_{\mu a}\partial^\mu u_b\tilde{u}_c: + \tilde{f}^4_{abc} :A_{\mu a}A^\mu_b\partial_\nu A^\nu_c: + \tilde{f}^5_{abc} :\partial_\nu A^\nu_a u_b\tilde{u}_c:\big\}\,, \quad \tilde{f}^4_{abc} = \tilde{f}^4_{bac}\,,$$

where the f's are arbitrary real constants. The first order coupling term is then antisymmetric with respect to the conjugation K: $\tilde{T}_1^K = -\tilde{T}_1$.

Adding divergence terms to $\tilde{T}_1$ will not change the physics of the theory (see Subsec. 3.3), so that we always calculate modulo divergences in the following. We can modify $\tilde{T}_1$ by adding divergences $-\frac{ig}{4}(\tilde{f}^1_{abc} + \tilde{f}^1_{cba})\partial_\nu :A_{\mu a}A_{\nu b}A^\mu_c:$ and $-ig\tilde{f}^3_{abc}\partial_\mu :A^\mu_a u_b\tilde{u}_c:$ and arrive at the equivalent first order coupling

$$T_1 = ig\big\{f^1_{abc} :A_{\mu a}A_{\nu b}\partial^\nu A^\mu_c: + f^2_{abc} :A_{\mu a}u_b\partial^\mu\tilde{u}_c: + f^4_{abc} :A_{\mu a}A^\mu_b\partial_\nu A^\nu_c: + f^5_{abc} :\partial_\nu A^\nu_a u_b\tilde{u}_c:\big\}\,, \tag{36}$$

where

$$f^1_{abc} = -f^1_{cba}\,, \qquad f^4_{abc} = f^4_{bac}\,. \tag{37}$$

The gauge variation of T_1 is

$$\begin{aligned} d_Q T_1 = &-gf^1_{abc}\big\{:\partial_\mu u_a A_{\nu b}\partial^\nu A^\mu_c + A_{\mu a}\partial_\nu u_b\partial^\nu A^\mu_c + A_{\mu a}A_{\nu b}\partial^\nu\partial^\mu u_c:\big\} \\ &- gf^2_{abc}\big\{:\partial_\mu u_a u_b\partial^\mu\tilde{u}_c + A_{\mu a}u_b\partial^\mu\partial_\nu A^\nu_c:\big\} \\ &- 2gf^4_{abc} :\partial_\mu u_a A^\mu_b\partial_\nu A^\nu_c: - gf^5_{abc} :\partial_\nu A^\nu_a u_b\partial_\mu A^\mu_c:. \end{aligned} \tag{38}$$

Gauge invariance requires that the gauge variation be a divergence:

$$\begin{aligned} d_Q T_1 = g\partial_\mu\big\{&g^1_{abc} :\partial^\mu u_a A_{\nu b}A^\nu_c: + g^2_{abc} :u_a A^\mu_b\partial_\nu A^\nu_c: \\ &+ g^3_{abc} :\partial^\mu u_a u_b\tilde{u}_c: + g^4_{abc} :u_a u_b\partial^\mu\tilde{u}_c: \\ &+ g^5_{abc} :\partial_\nu u_a A^\nu_b A^\mu_c: + g^6_{abc} :u_a\partial^\mu A^\nu_b A_{\nu c}: + g^7_{abc} :u_a\partial^\nu A^\mu_b A_{\nu c}:\big\}\,, \end{aligned} \tag{39}$$

where $g^1_{abc} = g^1_{acb}$, $g^4_{abc} = -g^4_{bac}$. Comparing the terms in (38) and (39), we immediately obtain the following set of constraints for the coupling coefficients:

$$-f^1_{cab} = 2g^1_{abc} + g^6_{abc} \quad \text{from} \quad :\partial^\mu u\partial_\mu A_\nu A^\nu:\,, \tag{40}$$

$$g^2_{abc} + g^5_{abc} = -2f^4_{abc} \qquad :\partial^\mu u A_\mu\partial_\nu A^\nu:\,, \tag{41}$$

$$g^2_{abc} + g^2_{acb} = -f^5_{bac} - f^5_{cab} \qquad :u\partial_\nu A^\nu\partial_\mu A^\mu:\,, \tag{42}$$

$$-f^2_{abc} = g^3_{abc} + 2g^4_{abc} \qquad :\partial_\mu u u\partial^\mu\tilde{u}:\,, \tag{43}$$

$$g^3_{abc} = g^3_{bac} \qquad :\partial_\mu u\partial^\mu u\tilde{u}:\,, \tag{44}$$

$$-f^1_{cba} - f^1_{bca} = g^5_{abc} + g^5_{acb} \qquad :\partial_\mu \partial_\nu u A^\nu A^\mu: \ , \tag{45}$$

$$-f^1_{acb} = g^5_{abc} + g^7_{abc} \qquad :\partial_\mu u \partial_\nu A^\mu A^\nu: \ , \tag{46}$$

$$g^6_{abc} = -g^6_{acb} \qquad :u \partial_\mu A_\nu \partial^\mu A^\nu: \ , \tag{47}$$

$$-f^2_{cab} = g^2_{acb} + g^7_{abc} \qquad :u \partial_\mu \partial_\nu A^\mu A^\nu: \ , \tag{48}$$

$$g^7_{abc} = -g^7_{acb} \qquad :u \partial_\nu A^\mu \partial^\mu A^\nu: \ . \tag{49}$$

From (45), (46) and (49) we immediately derive

$$f^1_{abc} + f^1_{acb} - f^1_{bca} - f^1_{cba} = 0 \, . \tag{50}$$

Combining this with (37), we obtain the first important result that f^1_{abc} is totally antisymmetric. g^1_{abc} is symmetric in b and c; from (47) and (40) we conclude $g^1_{abc} = 0$ and $g^6_{abc} = -f^1_{abc}$.

The ansatz (39) we used for $d_Q T_1$ is still too general, because it has to fulfill $d_Q^2 T_1 = 0$. Calculating $d_Q^2 T_1$ and setting the coefficients of all Wick monomials equal to zero we get the relations

$$\begin{aligned} 2g^1_{abc} - g^5_{bac} - g^5_{bca} + g^5_{abc} + g^6_{abc} + g^7_{abc} &= 0 \, , \\ 2g^1_{abc} - g^5_{cba} + g^6_{abc} - g^7_{cba} &= 0 \, , \\ g^2_{abc} + g^2_{bac} + g^3_{abc} + g^3_{bac} + g^5_{abc} + g^5_{bac} &= 0 \, , \\ g^2_{abc} - g^3_{bac} - g^4_{bac} + g^4_{abc} + g^7_{acb} &= 0 \, , \\ g^6_{abc} + g^6_{acb} + g^7_{abc} + g^7_{acb} &= 0 \, . \end{aligned} \tag{51}$$

Combining these equations with the previous ones and taking all symmetry properties of the f's and g's into account, we find $f^2_{abc} = -f^1_{abc}$, $f^4 = 0$, $f^5_{abc} = -f^5_{cba}$ (46)

$$\begin{aligned} &g^1 = 0 \, , \qquad g^2_{abc} = -g^5_{abc} \, , \qquad g^3 = 0 \, , \\ &g^4_{abc} = \frac{1}{2} f^1_{abc} \, , \quad g^6_{abc} = -f^1_{abc} \, , \quad g^7_{abc} = f^1_{abc} - g^5_{abc} \, . \end{aligned} \tag{52}$$

Then T_1 assumes the following form

$$T_1 = T_1^{\mathrm{YM}} + T_1^D \, , \tag{53}$$

$$T_1^{\mathrm{YM}} = igf^1_{abc} \left\{ :\frac{1}{2} A_{\mu a} A_{\nu b} F_c^{\nu\mu}: - :A_{\mu a} u_b \partial^\mu \tilde{u}_c: \right\} , \tag{54}$$

$$T_1^D = igf^5_{abc} :\partial_\nu A_a^\nu u_b \tilde{u}_c: , \tag{55}$$

where "YM" stand tentatively for "Yang–Mills" and "D" for "Deformation," and $f^5_{abc} = -f^5_{cba}$.

3.2. *Second order gauge invariance*

In this section we investigate the consequences of second order gauge invariance for the theory given by (54)

$$T_1 = igf_{abc}\Big\{:\frac{1}{2}A_{\mu a}A_{\nu b}F_c^{\nu\mu}: - :A_{\mu a}u_b\partial^\mu\tilde{u}_c:\Big\} + ig\tilde{f}_{abc}:\partial_\nu A_a^\nu u_b\tilde{u}_c:\,, \tag{56}$$

where f is totally antisymmetric and $\tilde{f}_{abc}$ antisymmetric in a and c. If we consider the product

$$T_1(x)T_1(y)\,, \tag{57}$$

then its gauge variation is simply given by

$$\begin{aligned} d_Q[T_1(x)T_1(y)] &= d_QT_1(x)T_1(y) + T_1(x)d_QT_1(y) \\ &= i\partial_\mu^x T_{1/1}^\mu(x)T_1(y) + iT_1(x)\partial_\mu^y T_{1/1}(y)\,. \end{aligned} \tag{58}$$

But the analogous result

$$d_QT_2(x,y) = i\partial_\mu^x T_{2/1}^\mu(x,y) + i\partial_\mu^y T_{2/2}^\mu(x,y) \tag{59}$$

is not automatically true. This can be seen as follows:

$$i\partial_\mu^x T_{1/1}^\mu(x)T_1(y) + (x\leftrightarrow y) \tag{60}$$

contains terms

$$\begin{aligned} &ig^2\partial_\mu f_{abc}f_{a'b'c'}:u_aA_{\nu b}\partial^\mu A_c^\nu(x)::A_{\mu' a'}A_{\nu' b'}\partial^{\nu'}A_{c'}^{\mu'}(y): + (x\leftrightarrow y) \\ &\quad = -g^2\partial_\mu^x\big\{f_{abc}f_{cb'c'}:u_a(x)A_{\nu b}(x)A_{\lambda b'}(y)\partial^\lambda A_{c'}^\nu(y):\partial_x^\mu D^+(x-y) \\ &\qquad + f_{abc}f_{a'cc'}:u_a(x)A_{\nu b}(x)A_{\lambda a'}(y)\partial^\nu A_{c'}^\lambda(y):\partial_x^\mu D^+(x-y) \\ &\qquad - f_{abc}f_{a'b'c}:u_a(x)A_{\nu b}(x)A_{a'}^\nu(y)A_{b'}^\lambda(y):\partial_x^\mu\partial_x^\lambda D^+(x-y)\big\} + (x\leftrightarrow y) + \cdots\,. \end{aligned} \tag{61}$$

If we replace now the distribution $D^+(x-y)$ simply by the Feynman propagator (time ordering of expression (61)), we obtain additional local terms ("anomalies"), because D_F satisfies the inhomogeneous wave equation $\square_x D_F(x-y) = \delta(x-y)$

$$\begin{aligned} A = -g^2\big\{&f_{abc}f_{cb'c'}:u_a(x)A_{\nu b}(x)A_{\lambda b'}(y)\partial^\lambda A_{c'}^\nu(y):\delta(x-y) \\ &+ f_{abc}f_{a'cc'}:u_a(x)A_{\nu b}(x)A_{\lambda a'}(y)\partial^\nu A_{c'}^\lambda(y):\delta(x-y) \\ &- f_{abc}f_{a'b'c}:u_a(x)A_{\nu b}(x)A_{a'}^\nu(y)A_{b'}^\lambda(y):\partial_\lambda^x\delta(x-y)\big\} + (x\leftrightarrow y) + \cdots\,. \end{aligned} \tag{62}$$

Using the distributional identity

$$\begin{aligned} &A(x)B(y)\partial_\mu^x\delta(x-y) + A(y)B(x)\partial_\mu^y\delta(x-y) \\ &\quad = A(x)\partial_\mu B(x)\delta(x-y) - \partial_\mu A(x)B(x)\delta(x-y)\,, \end{aligned} \tag{63}$$

we can rewrite A

$$\begin{aligned} A = -g^2\big\{&2f_{abc}f_{cb'c'} :u_a(x)A_{\nu b}(x)A_{\lambda b'}(y)\partial^\lambda A^\nu_{c'}(y): \delta(x-y) \\ &+ 2f_{abc}f_{a'cc'} :u_a(x)A_{\nu b}(x)A_{\lambda a'}(y)\partial^\nu A^\lambda_{c'}(y): \delta(x-y) \\ &+ f_{abc}f_{a'b'c} :u_a(x)\partial_\lambda A_{\nu b}(x)A^\nu_{a'}(y)A^\lambda_{b'}(y): \delta(x-y) \\ &+ f_{abc}f_{a'b'c} :\partial_\lambda u_a(x)A_{\nu b}(x)A^\nu_{a'}(y)A^\lambda_{b'}(y): \delta(x-y) \\ &- f_{abc}f_{a'b'c} :u_a(x)A_{\nu b}(x)\partial_\lambda A^\nu_{a'}(y)A^\lambda_{b'}(y): \delta(x-y) \\ &- f_{abc}f_{a'b'c} :u_a(x)A_{\nu b}(x)A^\nu_{a'}(y)\partial_\lambda A^\lambda_{b'}(y): \delta(x-y)\big\} + \cdots . \end{aligned} \tag{64}$$

Now we are free to add to $\partial^x_\lambda T^\lambda_{2/1}(x,y) + (x \leftrightarrow y)$ a normalization term

$$\begin{aligned} -g^2 f_{abc}f_{a'b'c}\partial^x_\lambda\{&:u_a(x)A_{\nu b}(x)A^\nu_{a'}(y)A^\lambda_{b'}(y): \delta(x-y)\} + (x \leftrightarrow y) \\ = -g^2 f_{abc}f_{a'b'c} &:\partial_\lambda u_a A_{\nu b}A^\nu_{a'}A^\lambda_{b'} + u_a\partial_\lambda A_{\nu b}A^\nu_{a'}A^\lambda_{b'} \\ &+ u_a A_{\nu b}\partial_\lambda A^\nu_{a'}A^\lambda_{b'} + \partial_\lambda u_a A_{\nu b}A^\nu_{a'}\partial_\lambda A^\lambda_{b'}: \ \delta(x-y) . \end{aligned} \tag{65}$$

Then the renormalized anomaly term A' is given by

$$\begin{aligned} A' = -g^2\big\{&2f_{abc}f_{cb'c'} :u_a(x)A_{\nu b}(x)A_{\lambda b'}(y)\partial^\lambda A^\nu_{c'}(y): \delta(x-y) \\ &+ 2f_{abc}f_{a'cc'} :u_a(x)A_{\nu b}(x)A_{\lambda a'}(y)\partial^\nu A^\lambda_{c'}(y): \delta(x-y) \\ &+ 2f_{abc}f_{a'b'c} :u_a(x)\partial_\lambda A_{\nu b}(x)A^\nu_{a'}(y)A^\lambda_{b'}(y): \delta(x-y) \\ &+ 2f_{abc}f_{a'b'c} :\partial_\lambda u_a(x)A_{\nu b}(x)A^\nu_{a'}(y)A^\lambda_{b'}(y): \delta(x-y)\big\} . \end{aligned} \tag{66}$$

Furthermore, we must add a normalization term

$$N = i\frac{g^2}{2} f_{abc}f_{a'b'c'} :A_{\lambda a}A_{\nu b}A^\nu_{a'}A^\lambda_{b'}: \delta(x-y) \tag{67}$$

to the gluon–gluon scattering term in $T_2(x,y)$ which contains a propagator $\sim \partial^\mu\partial^\nu D_F(x-y)$. Then the gauge variation of N

$$d_Q N = -2g^2 f_{abc}f_{a'b'c} :\partial_\lambda u_a A_{\nu b}A^\nu_{a'}A^\lambda_{b'}: \delta(x-y) \tag{68}$$

cancels the term in (66) containing the operator $\partial_\lambda u_a$. The remaining anomalies cancel, iff

$$f_{abc}f_{dec} + f_{adc}f_{ebc} + f_{aec}f_{bdc} = 0 , \tag{69}$$

i.e. if the structure constants obey the *Jacobi identity*. Consequently, the Lie algebra structure is obtained as a consequence of perturbative gauge invariance at first and second order.

3.3. *Analysis of the deformation*

Now we want to analyze the physical consequences of the various coupling terms in (56) and (55). The last term in (56) or (54) is a coboundary because $\tilde{f}_{abc}$ is antisymmetric in a and c:

$$\tilde{f}_{abc} :\partial_\nu A_a^\nu u_b \tilde{u}_c : = \frac{i}{2} d_Q(\tilde{f}_{abc} :\tilde{u}_a u_b \tilde{u}_c :) \,. \tag{70}$$

Summing up we have shown that a gauge invariant coupling of gauge and ghost fields is necessarily equal to the usual Yang–Mills form plus a coboundary term. Now the question arises whether the latter has physical consequences. Here the following result is relevant:[16]

Theorem (Conjecture). *Let P be the projection operator on the physical subspace* (22) *and $T_n^0(x_1, \ldots, x_n)$ the n-point distribution for the usual Yang–Mills theory* (*without divergence- and coboundary-couplings*)*. If $T_n(x_1, \ldots, x_n)$ is the n-point distribution of the general theory* (*with divergence- and coboundary couplings*), *then*

$$PT_n(x_1, \ldots, x_n)P = PT_n^0(x_1, \ldots, x_n)P + (\text{sum of divergences})\,. \tag{71}$$

Unfortunately, the proof is not complete in the case of coboundary couplings for arbitrary n (it has been proven for $n \le 4$). But there is no doubt that the theorem is true in general.

The sum of divergences in (71) have no physical effect because they would vanish in the formal adiabatic limit. Then (71) means that the physical S matrix elements are unchanged.

4. Massive Gauge Theories

If some asymptotic fields are massive, their contribution to the gauge charge (13) is changed according to

$$Q := \int d^3x \big(\partial_\mu A_a^\mu(x) + m_a \Phi_a(x)\big) \overleftrightarrow{\partial_0} \, u_a(x) \,. \tag{72}$$

Here $\Phi_a(x)$ are scalar bosonic fields which are quantized according to

$$(\Box + m_a^2)\Phi_a(x) = 0 \,, \qquad [\Phi_a(x), \Phi_b(y)] = -i\delta_{ab} D_{m_a}(x-y) \,. \tag{73}$$

This restores the nilpotency of Q. The gauge variations (20) are altered as follows

$$[Q, \Phi_a] = i m_a u_a \,, \qquad \{Q, \tilde{u}_a\} = -i(\partial_\mu A_a^\mu + m_a \Phi_a) \,. \tag{74}$$

The scalar fields Φ_a appearing in Q do not belong to the physical subspace (22).

The starting point is now a general ansatz of the form

$$T_1 = T_1^A + T_1^u + T_1^\Phi \,, \tag{75}$$

where $T_1^A + T_1^u$ is the same as in (36) and T_1^Φ is the most general coupling of Φ_a and gauge and ghost fields. The discussion of Subsec. 3.1 can be repeated, the only change are mass terms $\sim m_a$ or $\sim m_a^2$ which come from the Klein–Gordon equation

(73) and from (74). The latter must be compensated by means of T_1^Φ. This part of the gauge invariance problem is considered in Ref. 3 for the case of the electroweak theory. The discussion of second order gauge invariance in the Yang–Mills part (Subsec. 3.2) remains unchanged because all anomalies of Subsec. 3.2 occur in the bigger theory as well. But there are many more anomalies coming from T_1^Φ. It turns out that it is impossible to remove all those anomalies without enlarging the theory still further by introducing another (now physical) scalar field. In this way the Higgs field enters the scene. For the detailed discussion we refer to Refs. 3 and 4.

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