Finite Field Theories
and
Causality

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Overwiew

Naive example of a 'UV divergence'

Consider Heaviside-Θ- and Dirac-δ-distributions in 1-dim. 'configuration space':

Product $\Theta(x)\delta(x)$ obviously ill-defined !

Fourier transforms
$$\mathcal{F}\{\delta\}(k) = \hat{\delta}(k) = \int dx\, \delta(x) e^{-ikx} = 1,$$
$$\hat{\Theta}(k) = \lim_{\epsilon \searrow 0} \int dx\, \Theta(x) e^{-ikx - \epsilon x} = \lim_{\epsilon \searrow 0} \left. \frac{i e^{-ikx-\epsilon x}}{k - i\epsilon} \right|_0^\infty = -\frac{i}{k - i0}.$$

Calculate nevertheless the Fourier transform of the ill-defined product
$$\mathcal{F}\{\Theta\delta\}(k) = \int dx\, e^{-ikx} \Theta(x)\delta(x) = \int dx\, e^{-ikx} \int \frac{dk'}{2\pi} \hat{\Theta}(k') e^{+ik'x} \int \frac{dk''}{2\pi} \hat{\delta}(k'') e^{+ik''x}$$

Since $\int dx\, e^{i(k'+k''-k)x} = 2\pi\delta(k'+k''-k)$, we obtain the convolution
$$\mathcal{F}\{\Theta\delta\}(k) = \frac{1}{2\pi} \int dk'\, \hat{\Theta}(k') \hat{\delta}(k-k') = -\frac{i}{2\pi} \int \frac{dk'}{k' - i0}$$

Obvious problem in x-space $\rightarrow$ divergent integral in k-space !

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Interpretation of UV divergences

In perturbative quantum field theory, the rôle of the Heaviside Θ-distribution is taken by over by the time-ordering operator.

'Textbook' expression for the perturbative scattering matrix:

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{+\infty} dt_1 \ldots \int_{-\infty}^{+\infty} dt_n \, T[H_{int}(t_1) \ldots H_{int}(t_n)]$$

$$= \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \ldots \int d^4x_n \, T[\mathcal{H}_{int}(x_1) \ldots \mathcal{H}_{int}(x_n)].$$

$H_{int}(t)$ $\{\mathcal{H}_{int}(x)\}$: Interaction Hamiltonian {density}. $H_{int}(t) = \int d^3x \, \mathcal{H}_{int}(x)$.

A time-ordered expression à la

$$T[\mathcal{H}_{int}(x_1) \ldots \mathcal{H}_{int}(x_n)] = \sum_{Perm.\ \Pi} \Theta(x^0_{\Pi_1} - x^0_{\Pi_2}) \ldots \Theta(x^0_{\Pi_{(n-1)}} - x^0_{\Pi_n}) \mathcal{H}_{int}(x_{\Pi_1}) \ldots \mathcal{H}_{int}(x_{\Pi_n})$$

is formal (ill-defined) ! Products of $\mathcal{H}_{int}$ too singular to be multiplied by Θ-distributions.

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Historical details

E. C. G. Stückelberg (∼1949)

- Construction of the S-matrix by means of causality and unitarity.

N. N. Bolgolyubov et al. (∼1959)

- Reformulation of the causality condition (see below).
- Adiabatic switching (see below).
- UV divergences persist.

H. Epstein & V. Glaser
(Annales de l'institut Henri Poincaré (A), Physique théorique, 19 (211-295) 1973)

- *Inductive* construction of the perturbation series by means of Poincaré invariance and causality (unitarity plays no immediate rôle).

  → no UV divergences
  → Feynman rules only hold on tree-level
  → Loop diagrams rather technical
  → Finite dispersion integrals instead of divergent Feynman integrals
  → New strategy to treat the infrared problem by adiabatic switching of interaction
  → Only scalar field theory

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Free quantum fields

Crucial observation: Free field operators are operator-valued distributions.
E.g.: The scalar (neutral) field with mass m

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3k}{\sqrt{2E}} \left[a(\vec{k}) e^{-ikx} + a^{+}(\vec{k}) e^{+ikx} \right], \quad E = \sqrt{\vec{k}^2 + m^2},$$

must be smeared out by rapidly decreasing test functions $g(x)$ in the Schwartz space $\mathcal{S}(\mathbb{R}^4)$ ($\mathcal{S}(\mathbb{R}^3)$) in order to get an operator in Fock space, formally

$$\varphi(g) = \int d^4x \, \varphi(x) g(x).$$

$\varphi(x)|0\rangle$ is not a Fock state !

The same arguing applies, e.g., to the interaction Hamiltonian densities used in perturbation theory constructed from normally ordered products of free fields:

- QED: Spinor field $\Psi(x)$, photon field $A_\mu(x) \to \mathcal{H}_{int} = -e : \bar{\Psi}(x)\gamma^\mu \Psi(x) : A_\mu(x)$.
- φ^3-Theory $\to \mathcal{H}_{int} = \frac{\lambda}{3!} : \varphi(x)^3 :$.
- ...

Perturbative S-matrix

It is therefore most natural to replace the problematic expression

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int d^4x_1 \dots d^4x_n \, T[\mathcal{H}_{int}(x_1) \dots \mathcal{H}_{int}(x_n)] \quad \text{by}$$

$$S(g) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n \, T_n(x_1, \dots, x_n) g(x_1) \dots g(x_n), \quad g \in \mathcal{S}(\mathbb{R}^4),$$

where $g(x)$ is a locally varying coupling constant and $T_1(x) = -i\mathcal{H}_{int}(x)$.

$T_n(x_1, \dots, x_n) \simeq T[T_1(x_1) \dots T_1(x_n)]$

is a well-defined (divergence free) time-ordered product with

$$T_n(\dots, x_i, \dots, x_j, \dots) = T_n(\dots, x_j, \dots, x_i, \dots) \;\; \forall i, j$$

by construction.

Adiabatic limit

In the causal (Epstein-Glaser-) approach, the T_n are free of any UV divergences.

$$S_n(g) = \frac{1}{n!} \int d^4x_1 \dots d^4x_n \, T_n(x_1, \dots, x_n) g(x_1) \dots g(x_n)$$

is well-defined at every order n of the perturbation expansion (even when massless fields are present).

$\longrightarrow$ No statements about the convergence of the full series !

$\longrightarrow$ Infrared problems arise in the adiabatic limit $g(x) \to 1$, since $1 \notin \mathcal{S}(\mathbb{R}^4)$.

$\longrightarrow$ Performing the adiabatic limit is a delicate task ($\longrightarrow$ existence and uniqueness !): Limit has to be taken such that observable quantities (cross sections) remain finite.

$\longrightarrow$ Typical approach: Rescaling $g(x)$ according to

$$\lim_{\epsilon \searrow 0} g(\epsilon x) \;\; " \longrightarrow " \;\; g(x) = g(0) = const. \sim coupling \;\; constant.$$

$\longrightarrow$ No further regularizations necessary ($\longrightarrow$ finite photon mass).

Formal inversion of the perturbative S-matrix

The perturbative S-matrix

$$S(g) = \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n\, T_n(x_1, \ldots, x_n) g(x_1) \ldots g(x_n) = \mathbb{1} + T$$

can be formally inverted

$$S(g)^{-1} = \mathbb{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \ldots d^4x_n\, \tilde{T}_n(x_1, \ldots, x_n) g(x_1) \ldots g(x_n)$$

$$= (\mathbb{1} + T)^{-1} = \mathbb{1} + \sum_{r=1}^{\infty} (-T)^r .$$

$$\longrightarrow \tilde{T}_n(X) = \sum_{r=1}^{n} (-1)^r \sum_{P_r} T_{n_1}(X_1) \ldots T_{n_r}(X_r),$$

where $X = \{x_1, \ldots, x_n\}$ is a disordered set and $\sum_{P_r}$ denotes all partitions of X into r disjoint subsets

$$X = X_1 \cup \ldots \cup X_r, \quad X_j \neq \emptyset, \quad X_i \cap X_j = \emptyset, \quad |X_j| = n_j.$$

Causality (the pivotal point)

We assume that $g(x) = g_1(x) + g_2(x)$ can be decomposed such that the supports of $g_1(x)$ and $g_2(x)$ are space-like separated, i.e.

$\exists$ reference frame such that $x \in \text{supp}(g_1) \Rightarrow x^0 < 0$ and $y \in \text{supp}(g_2) \Rightarrow y^0 > 0$.

Causality condition

$$S(g_1 + g_2) = S(g_2)S(g_1)$$

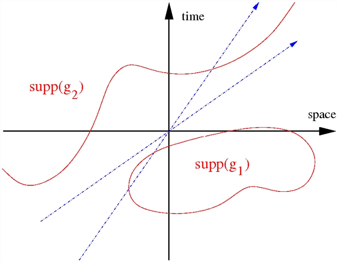
$$\forall\, g_1, g_2 \text{ where } supp(g_1) < supp(g_2).$$

This implies

$$T_n(x_1, \ldots, x_n) = T_m(x_1, \ldots, x_m)T_{n-m}(x_{m+1}, \ldots, x_n)$$

if $\{x_1, \ldots, x_m\} > \{x_{m+1}, \ldots, x_n\}$.

This condition is, of course, intuitively clear.



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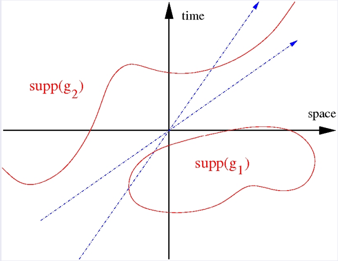

$supp(g_2) > supp(g_1)$

Explicit construction of $S_2(g)$

Causality and translation invariance requires that the commutator $D_2(z = x_1 - x_2)$

$$D_2(x_1 - x_2) = (-i)^2[\mathcal{H}_{int}(x_1), \mathcal{H}_{int}(x_2)] = [T_1(x_1), T_1(x_2)] = 0 \quad \text{for} \quad (x_1 - x_2)^2 < 0$$

has causal support on the closed light-cone $V = \bar{V}^+ \cup \bar{V}^-$ in the sense of distributions. We introduce (primed) advanced and retarded distributions $A_2'(z)$ and $R_2'(z)$:

Splitting of D_2

$$R_2 = +D_2\big|_{\bar{V}^+ - \{0\}} \quad , \quad A_2 = -D_2\big|_{\bar{V}^- - \{0\}},$$

$$R_2' = -T_1(x_2)T_1(x_1) \quad , \quad A_2' = -T_1(x_1)T_1(x_2).$$

The non-trivial (!) splitting of D_2 corresponds to time-ordering:

$$T[T_1(x_1)T_1(x_2)] \text{"="} T_2(x_1, x_2) = R_2 - R_2' = A_2 - A_2'.$$

Test: $R_2 - R_2'$ for $z^0 > 0$ and $z^0 < 0$.

$z^0 > 0 \quad : \quad T_1(x_1)T_1(x_2) - T_1(x_2)T_1(x_1) + T_1(x_2)T_1(x_1)$

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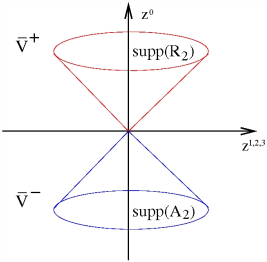

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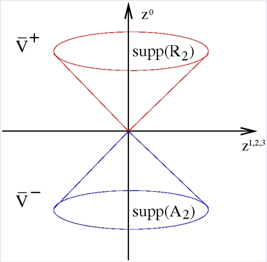

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φ^3 – *Theory*

With $T_1(x) = \frac{i\lambda}{3!} : \varphi(x)^3 :$ and $\langle 0| T(\varphi(x_1)\varphi(x_2))|0\rangle = i\Delta_F(x_1 - x_2)$,
standard Wick ordering leads to

$$
\begin{aligned}
T_2(x_1, x_2) \quad " = " \quad & -\frac{\lambda^2}{3!^2} : \varphi(x_1)^3\varphi(x_2)^3 : -\frac{9\lambda^2}{3!^2} : \varphi(x_1)^2\varphi(x_2)^2 : i\Delta_F(x_1 - x_2) \\
& -\frac{18\lambda^2}{3!^2} : \varphi(x_1)\varphi(x_2) : [i\Delta_F(x_1 - x_2)]^2 - \frac{\lambda^2}{3!}[i\Delta_F(x_1 - x_2)]^3 .
\end{aligned}
$$

In the causal approach, one constructs first

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$$

where $\Delta(x_1 - x_2)$ is the Pauli-Jordan distribution, which can be decomposed into the positive- and negative-frequency Pauli-Jordan distributions

$$\Delta(z) = \Delta^+(z) + \Delta^-(z),$$

$$\Delta^{\pm}(z) = \mp\frac{i}{(2\pi)^3}\int d^4k\,\Theta(\pm k^0)\delta(k^2 - m^2)e^{-ikx}.$$

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Splitting of D_2 (tree level)

In order to get the retarded C-number part $D_2^{tree}(z) := \Delta(z)$, we simply multiply by $\Theta(z^0)$: $R_2^{tree}(z) = \Theta(z^0)\Delta(z)$.

From $\hat{\Theta}(k) = \frac{i(2\pi)^3}{k^0+i0}\delta^{(3)}(\vec{k})$ follows

$$\hat{R}_2^{tree}((k^0,\vec{0})) = \frac{i}{2\pi}\int dp^0 \frac{\hat{\Delta}((p^0,\vec{0}))}{k^0 - p^0 + i0} = \frac{i}{2\pi}\int dt \frac{\hat{\Delta}((tk^0,\vec{0}))}{1 - t + i0},$$

or, from the Lorentz covariance of Δ ($\hat{\Delta}$) we obtain a dispersion relation

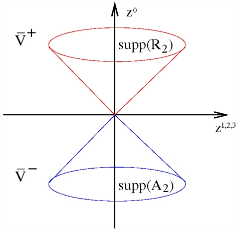
$$\hat{R}_2^{tree}(k) = \frac{i}{2\pi}\int dt \frac{\hat{\Delta}(tk)}{1 - t + i0} \quad \text{for } k^0 \in V^+.$$

Splitting of D_2^{tree}

$$\hat{\Delta}(k) = -2\pi i \,\mathrm{sgn}(k^0)\delta(k^2 - m^2) \quad \rightarrow$$

$$\hat{R}_2^{tree}(k) = \int dt \frac{\mathrm{sgn}(tk^0)\delta(t^2k^2 - m^2)}{1 - t + i0} =$$

$$\int dt \frac{[\delta(t - \frac{m}{\sqrt{k^2}}) - \delta(t + \frac{m}{\sqrt{k^2}})]}{2\sqrt{k^2}m(1 - t + i0)} = \frac{1}{k^2 - m^2}.$$



$$\hat{R}_2^{tree}(k) = \frac{1}{(2\pi)^4}\hat{D}_2(k) * \mathcal{F}\{\Theta(z^0)\}(k)$$

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In order to get the retarded C-number part $D_2^{tree}(z) := \Delta(z)$, we simply multiply by $\Theta(z^0)$: $R_2^{tree}(z) = \Theta(z^0)\Delta(z)$.

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$$\hat{R}_2^{tree}((k^0,\vec{0})) = \frac{i}{2\pi}\int dp^0 \frac{\hat{\Delta}((p^0,\vec{0}))}{k^0 - p^0 + i0} = \frac{i}{2\pi}\int dt \frac{\hat{\Delta}((tk^0,\vec{0}))}{1 - t + i0},$$

or, from the Lorentz covariance of Δ ($\hat{\Delta}$) we obtain a dispersion relation

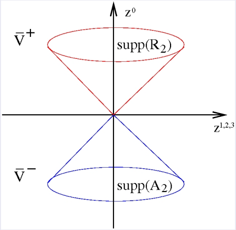
$$\hat{R}_2^{tree}(k) = \frac{i}{2\pi}\int dt \frac{\hat{\Delta}(tk)}{1 - t + i0} \quad \text{for } k^0 \in V^+.$$

Splitting of D_2^{tree}

$$\hat{\Delta}(k) = -2\pi i \,\mathrm{sgn}(k^0)\delta(k^2 - m^2) \quad \rightarrow$$

$$\hat{R}_2^{tree}(k) = \int dt \frac{\mathrm{sgn}(tk^0)\delta(t^2k^2 - m^2)}{1 - t + i0} =$$

$$\int dt \frac{[\delta(t - \frac{m}{\sqrt{k^2}}) - \delta(t + \frac{m}{\sqrt{k^2}})]}{2\sqrt{k^2}m(1 - t + i0)} = \frac{1}{k^2 - m^2}.$$



$$\hat{R}_2^{tree}(k) = \frac{1}{(2\pi)^4}\hat{D}_2(k) * \mathcal{F}\{\Theta(z^0)\}(k)$$

Splitting of D_2 (loop level)

The self-energy (one-loop) part in T_2 is logarithmically divergent:

$$T_2^{loop}(x_1 - x_2) \sim [i\Delta_F(x_1 - x_2)]^2 \xrightarrow{\mathcal{F}} \int \frac{d^4p}{[p^2 - m^2 + i0][(k-p)^2 - m^2 + i0]} .$$

In the causal approach, one calculates first $D_2(x_1 - x_2) = [T_1(x_1), T_1(x_2)]$, leading to

$$D_2^{loop}(x_1 - x_2) \sim [\Delta^-(x_1 - x_2)]^2 - [\Delta^-(x_2 - x_1)]^2 \xrightarrow{\mathcal{F}} \mathrm{sgn}(k^0)\Theta(k^2 - 4m^2).$$

Naive splitting also leads to a divergent dispersion integral:

$$\Theta(z^0) D_2^{loop}(z) \xrightarrow{\mathcal{F}} \int dt \, \frac{\hat{D}_2^{loop}(tk)}{1 - t + i0} \quad (k \in V^+).$$

However, it can be shown that the retarded part can be obtained from a subtracted dispersion integral ($m \neq 0$)

$$\hat{R}_2^{loop}(k) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\hat{D}_2^{loop}(tk)}{(t - i0)^{\omega+1}(1 - t + i0)} + const. \quad (k \in V^+) \quad \text{with } \omega = 0.$$

ω depends on the scaling properties ($\longrightarrow$ power counting) of the causal distribution D_2^{loop}.

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Strategy

With causality as the fundamental input, it is possible to construct causal distributions

$$A_n,\ R_n \text{ and } D_n(x_1,\dots,x_n) = R_n(x_1,\dots,x_n) - A_n(x_1,\dots,x_n)$$

at higher orders:

$$\begin{aligned} \operatorname{supp} R_n(x_1,\dots,x_n) &\subseteq \Gamma^+(x_n), \\ \operatorname{supp} A_n(x_1,\dots,x_n) &\subseteq \Gamma^-(x_n), \end{aligned}$$

where

$$\Gamma^{\pm}(x_n) = \{(x_1,\dots,x_n)\,|\,x_j \in \bar{V}^{\pm}(x_n)\,\forall\, j = 1,\dots,n-1\},$$

$$\bar{V}^+(x) = \{y\,|\,(y-x)^2 \geq 0,\ y^0 \geq x^0\},\quad \bar{V}^-(x) = \{y\,|\,(y-x)^2 \geq 0,\ y^0 \leq x^0\},$$

and

$$\operatorname{supp} D_n(x_1,\dots,x_n) \subseteq \Gamma^+(x_n) \cup \Gamma^-(x_n).$$

Recipe

T_m known for $1 \leq m \leq n-1$

$\Downarrow$

construct advanced/retarded distributions

$$A'_n(x_1,\ldots,x_n) = \sum_{P_2} \tilde{T}_{n_1}(X) T_{n-n_1}(Y,x_n) \quad \textit{and} \quad R'_n(x_1,\ldots,x_n) = \sum_{P_2} T_{n-n_1}(Y,x_n) \tilde{T}_{n_1}$$

with $P_2 : \{x_1,\ldots,x_{n-1}\} = X \cup Y,\ X \neq \emptyset, n_1 = |X| \geq 1$

$\Downarrow$

allow $X = \emptyset$

$$A_n(x_1,\ldots,x_n) = A'_n(x_1,\ldots,x_n) + T_n(x_1,\ldots,x_n)\,, \quad R_n(x_1,\ldots,x_n) = R'_n(x_1,\ldots,x_n) + T_n(x_1,\ldots,x_n)$$

$\Downarrow$

T_n unknown, but difference distribution

$$D_n = R'_n - A'_n = R_n - A_n$$

can be shown to be causal: splitting D_n generates R_n

$\Downarrow$

$$T_n = R_n - R'_n\,!$$

Splitting

The C-number parts of R_n and D_n

$$r^{tree,loop,\ldots}(x_1 - x_n, \ldots, x_{n-1} - x_n), \; d^{tree,loop,\ldots}(x_1 - x_n, \ldots, x_{n-1} - x_n)$$

go over into

$$\hat{r}^{tree,loop,\ldots}(p_1, \ldots, p_{n-1}) \text{ and } \hat{d}^{tree,loop,\ldots}(p_1, \ldots, p_{n-1})$$

via Fourier transformation.

If at least one field in the theory is massive, it can be shown that for $p = (p_1, p_2, \ldots) \in \Gamma^+$

$$\hat{r}(p) = \frac{i}{2\pi} \int_{-\infty}^{+\infty} \frac{\hat{d}(tp)}{(t - i0)^{\omega+1}(1 - t + i0)} dt + \sum_{|\alpha|=0}^{\omega \text{ or } \ldots} c_\alpha p^\alpha \quad (\alpha : \text{multi-index}),$$

where ω is a rigorously defined power-counting degree of divergence.

The terms $\sum c_\alpha p^\alpha$ correspond to the divergent parts of Feynman integrals in the standard regularization methods. They are due to the fact that the splitting of D_n into R_n and A_n is not uniquely defined at the "tip" $x_1 = x_2 = \ldots = x_n$ of the generalized forward/backward light-cones $\Gamma^\pm$.

In configuration space, they correspond to local terms $\sim \sum \hat{c}_\alpha D^\alpha \delta(x_1 - x_n, \ldots, x_{n-1} - x_n)$.

Perturbation theory does not specify them, and they have to be restricted, e.g., by symmetry considerations.

QCD

Considering QCD within the causal approach, it is most natural to start from a first order gluon field coupling (matter fields neglected)

$$T_1(x) = i\frac{g}{2} f_{abc} : A^a_\mu(x) A^b_\nu(x) F_c^{\nu\mu}(x) :, \quad F_c^{\mu\nu}(x) = \partial^\mu A_c^\nu(x) - \partial^\nu A_c^\mu(x).$$

First order gauge invariance requires additional fields (ghosts)

$$T_1(x) = igf_{abc} : A^a_\mu(x) A^b_\nu(x) \partial^\nu A_c^\mu(x) : - igf_{abc} : A_a^\mu(x) u_b(x) \partial^\mu \tilde{u}_c(x) : .$$

At second order, tree diagrams containing C-number distributions

$$\sim \partial_\mu \partial_\nu \Delta(x_1 - x_2) \xrightarrow{\mathcal{F}} -k^\mu k^\nu \hat{\Delta}(k)$$

appear; the splitting is fixed up to a local term $\sim g^{\mu\nu}\delta^{(4)}(x_1 - x_2)$

$$k^\mu k^\nu \hat{\Delta}(k) \xrightarrow{\text{splitting}} \frac{k^\mu k^\nu}{k^2 + i0} + Cg^{\mu\nu}.$$

Second order gauge invariance determines the constant C such that the usual four-gluon coupling term is generated.

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What has been done so far

- The finite/causal/Epstein-Glaser approach has been rediscovered by Michael Dütsch and Günter Scharf (U. Zürich) in 1985.
- Complete discussion of QED and introduction to the causal approach: "Finite Quantum Electrodynamics" by G. Scharf (1989, 2nd edition 1995).
- Zürich group, 1989-1992: Interacting fields, axial anomalies, full discussion of the renormalizability of scalar QED, ...
- 1992-1997: Complete discussion of perturbative QCD, gauge theories like the standard model (including spontaneous symmetry breaking) studied.
- since 1997: Quantum gravity and supersymmetric theories considered.
- Other groups: Klaus Fredenhagen et al. (causal approach on curved space-times)

$\longrightarrow$ Talk by Ernst Werner !

FIN