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Bound-free pair production cross-section in heavy-ion colliders from the equivalent photon approach

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Abstract – Exact calculations of the electron-positron pair production by a single photon in the Coulomb field of a nucleus with simultaneous capture of the electron into the K-shell are discussed for different nuclear charges. Using the equivalent photon method of Weizsäcker and Williams, a simple expression for the bound-free production of e^+e^- pairs by colliding very-high-energy fully stripped heavy ions is derived for nuclei of arbitrary charge.

Introduction. – There has been some theoretical and experimental interest in the process of single-photon pair production during the last few decades. Precise knowledge of the single-photon bound-free pair production cross-section in the electrostatic field of a fully stripped nucleus allows to calculate the bound-free pair production at the interaction point of a heavy-ion collider by using the equivalent photon method of Weizsäcker and Williams [1–3]. Recently, first observations of beam losses have been reported from measurements performed at the BNL Relativistic Heavy-Ion Collider (RHIC), where a beam of $^{63}\mathrm{Cu}^{29+}$ ions with 100 GeV/nucleon has been used [4]. In this paper, we derive a simple and compact expression for the bound-free pair production in relativistic heavy-ion collisions, which is expected to be a major luminosity limit for the Large Hadron Collider (LHC) when it operates with heavy ions because the localized energy deposition by the lost ions which have captured an electron may quench superconducting magnet coils [5].

The equivalent photon method. – The simplest way to get a reliable estimate of the cross-section in RHI collisions is provided by the equivalent photon method, which is originally due to Fermi [6], and later on developed by Weizsäcker and Williams. Within this framework, the target ion is considered as fixed. The projectile with impact parameter b is assumed to move on a straight line with velocity v and a relativistic Lorentz factor $\gamma_p = 2\gamma_c^2 - 1$, where γ_c is the Lorentz boost of the ions in the center-of-mass frame. The projectile is accompanied by its contracted electromagnetic field, which corresponds to a spectrum of equivalent photons, given by

$$N(\omega, b) = \frac{Z_p^2 \alpha}{\pi^2}\left(\frac{\omega^2}{\gamma_p^2 v^4}\right)\left[K_1^2(x) + \frac{1}{\gamma_p^2}K_0^2(x)\right], \qquad (1)$$

where $K_0(K_1)$ are the modified Bessel functions of the second kind of order zero (one), $x = \omega b/\gamma_p v$, and α is the fine-structure constant. The cross-section for an electromagnetic process in a highly relativistic collision is then obtained by integrating the single-quantum cross-section over the frequency spectrum and from a minimum impact parameter $b_{min} = R$, which was, in our case, chosen to be the Compton wavelength as the typical length scale for pair production, to infinity:

$$n(\omega) = \int\limits_R^\infty 2\pi b N(\omega, b)\mathrm{d}b, \qquad (2)$$

$$\sigma = \int n(\omega)\sigma_\gamma(\omega)\frac{\mathrm{d}\omega}{\omega}, \qquad (3)$$

where σ is now the total cross-section of the electromagnetic process. The integration of $N(\omega, b)$ over b can be carried out to give [7]

$$n(\omega) = 2\pi\int\limits_R^\infty bN(\omega, b)\mathrm{d}b =$$
$$\frac{2}{\pi}\frac{Z_p^2\alpha}{v^2}\left[\zeta K_0(\zeta)K_1(\zeta) - \frac{v^2\zeta^2}{2}(K_1^2(\zeta) - K_0^2(\zeta))\right], \qquad (4)$$

where $\zeta = \omega R/\gamma_p v$ is an adiabatic cutoff parameter. For $\gamma_p \gg 1$ (except for extreme low-energy frequencies, satisfying the relationship $\omega R/c \ll 1$), one can use the excellent approximation

$$n(\omega) = \frac{1}{\pi} Z_p^2 \alpha \log\left[\left(\frac{\delta}{\zeta}\right)^2 + 1\right] \cong \frac{2}{\pi} Z_p^2 \alpha \log\left(\frac{\delta}{\zeta}\right), \qquad (5)$$

where $\delta = 0.68108\ldots$ is a number related to Euler's constant. In the limit of very large frequencies, $\omega \gg \gamma_p v/R$, an adiabatic cutoff sets in and one has $n(\omega) \cong (\alpha Z_p^2/2) e^{-2\omega R/\gamma_p v}$. For practical calculations, it is usually sufficient to set the velocity v equal to the speed of light $c = 1$.

Calculations of the single-photon bound-free pair production cross-section $\sigma^K_{e^+e^-}$ with simultaneous capture of the electron into the K-shell of a target nucleus with nuclear charge number Z_t have been presented in [8]. In the meantime, the program used in [8] has been refined to higher precision, leading to small corrections of the originally calculated K-shell cross-sections of some few percent. In [9], it has been shown that the higher shells contribute about 20% of the K-shell capture to the total bound-free pair production cross-section $\sigma_{e^+e^-}$, *i.e.*, $\sigma_{e^+e^-} = \zeta(3)\sigma^K_{e^+e^-}$ provides a good approximation for $\sigma_{e^+e^-}$, where ζ is the Riemann zeta function defined by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \qquad (6)$$

for $x > 1$, and $\zeta(3) \simeq 1.2020569$.

Single-photon cross-sections. – Theoretical calculations using semirelativistic Sommerfeld-Maue wave functions for the positron and a semi-relativistic approximation for the bound electron wave functions valid for $\alpha Z_t \ll 1$ show that, for $\hbar\omega \gg 1$ and light nuclei, the cross-section $\sigma^K_{e^+e^-}$ is approximately given by the high-energy limit of the so-called Sauter cross-section [10]

$$\sigma^K_{e^+e^-} = 4\pi\lambda_c^2 \alpha^6 Z_t^5 \frac{m_e c^2}{\hbar\omega}, \qquad (7)$$

where m_e is the electron mass and λ_c the Compton wavelength.

The high-energy limit of the Sauter cross-section overestimates the exact cross-sections at low photon energies. Indeed, Sauter originally derived an exact result in the limit of vanishing Z_t for the K-shell photoelectric effect, which can be transcribed into the corresponding cross-section for pair creation [11] ($m_e = \hbar = c = 1$)

$$\sigma^{K(Sauter)}_{e^+e^-} = 4\pi\lambda_c^2\alpha^6 Z_t^5 \frac{{p_+}^3}{\omega^5} \times\left(\frac{E_+(E_+ + 2)}{E_+ - 1} - \frac{4}{3} - \frac{E_+ + 2}{2p_+(E_+ - 1)} \log\left[\frac{E_+ + p}{E_+ - p}\right]\right), \qquad (8)$$

where E_+ and $p_+ = (E_+^2 - 1)^{1/2}$ is the energy and the momentum of the positron, respectively. The photon threshold energy for pair production $\omega_{th} = 1 + \gamma_e$ is given by the sum of the positron rest energy and the energy $\gamma_e = (1 - \alpha^2 Z_t^2)^{1/2}$ of the bound electron in the K-shell. Generally, the positron energy is given by $E_+ = \omega - \gamma_e$, and in the limit $Z_t \to 0$ one obviously has $\omega = E_+ + 1$.

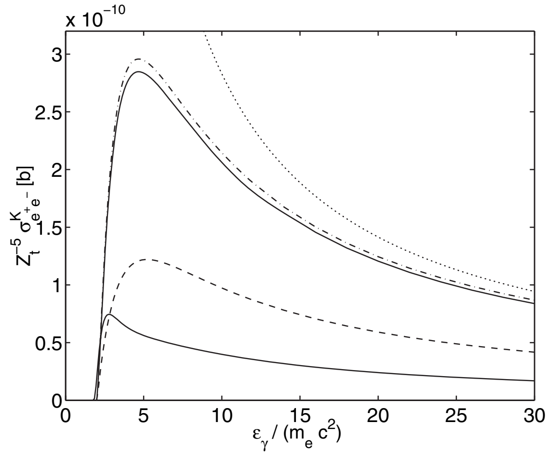

Fig. 1: Single-photon cross-section for pair production with K-shell capture as a function of the photon energy $\epsilon_\gamma = \hbar\omega$ for different nuclear charges Z_t. The cross-sections are scaled with Z_t^5. The dash-dotted curve displays the Sauter cross-section according to eq. (8), which agrees well with the numerical result for $Z_t = 1$ (upper solid curve). The dashed curve corresponds to $Z_t = 29$ (copper), the lower solid curve to $Z_t = 92$ (uranium). The dotted curve shows the high-energy limit of the Sauter cross-section according to eq. (7).

A comparison of the Sauter cross-section and exact numerical calculations for $Z_t = 1$, 29, and 92, is shown in fig. 1. The numerical result for $Z_t = 1$ agrees well with eq. (8). However, at high energies, the cross-section for hydrogen is smaller by a factor of 0.971 than the Sauter cross-section. This effect becomes more pronounced for target nuclei with large Z_t, such that eq. (7) must be modified by an additional factor $f(Z_t)$ according to

$$\sigma^K_{e^+e^-} = 4\pi\lambda_c^2\alpha^6 Z_t^5 f(Z_t)\frac{1}{\omega}. \qquad (9)$$

Some values for $[Z_t, f(Z_t)]$ obtained from exact numerical calculations are [92,0.196], [82,0.216], [29,0.484] and [1,0.971]. These values agree very well with those found by Pratt [12] and coincide with the values calculated by Agger and Sorensen [13].

A useful approximation of $f(Z_t)$ is given by

$$f(Z_t) = \left(\frac{53}{100} + \frac{2\alpha Z_t}{9}\right)^{2\pi\alpha Z_t}, \qquad (10)$$

which reproduces $f(Z_t)$ to an accuracy of 2% in the physically relevant range $0 \leqslant Z_t \leqslant 92$.

As a further effect one observes that the position and the shape of the cross-section peak varies for different nuclear charges. Whereas the peak is shifted to higher photon energies for light and medium-heavy nuclei, it is located at lower energies with respect to the Sauter cross-section peak for $Z_t \geqslant 62$.

Pair production in heavy-ion collisions. – Equation (8) in conjunction with the equivalent photon method allows to derive a precise numerical expression for the total bound-free pair production in relativistic heavy-ion collisions in the case of small target charge $\alpha Z_t \ll 1$ (with $R=1$ and $v=1$ in eq. (5)), given by

$$\sigma_{RHI} = \zeta(3)\sigma^K_{RHI} = \zeta(3)\int_2^\infty n(\omega)\sigma^{K(Sauter)}_{e^+e^-}(\omega)\frac{\mathrm{d}\omega}{\omega} \simeq$$
$$8\zeta(3)\lambda_c^2\alpha^7 Z_t^5 Z_p^2 \int_2^\infty \log\left(\frac{\delta\gamma_p}{\omega}\right)\sigma^{K(Sauter)}_{e^+e^-}(\omega)\frac{\mathrm{d}\omega}{\omega} =$$
$$8\zeta(3)\lambda_c^2\alpha^7 Z_t^5 Z_p^2[c_1\log\gamma_p - c_2], \tag{11}$$

with $c_1 = 0.21746$ and $c_2 = 0.58706$. Equation (11) still overestimates the cross-section, since the single-photon cross-section is suppressed by a factor $f(Z_t)$ at large photon energies. Additionally, the shape of the single-photon cross-section peak has some impact on σ_{RHI}, since the photon flux generated by the Coulomb field of the projectile becomes large at low equivalent photon energies. The defect can be remedied by a minor modification of eq. (11)

$$\sigma_{RHI} = 8\zeta(3)\lambda_c^2\alpha^7 Z_t^5 Z_p^2 f(Z_t)[c_1(Z_t)\log\gamma_p - c_2(Z_t)], \tag{12}$$

where a numerical fit for $0 \leqslant Z_t \leqslant 92$ gives

$$c_1(Z_t) \cong \left[1 - \frac{Z_t}{2.5\cdot 10^2} + \frac{Z_t^2}{1.5\cdot 10^4} - \frac{Z_t^3}{7.0\cdot 10^6}\right]c_1, \tag{13}$$

$$c_2(Z_t) \cong \left[1 - \frac{Z_t}{2.0\cdot 10^2} + \frac{Z_t^2}{1.2\cdot 10^4} - \frac{Z_t^3}{7.0\cdot 10^6}\right]c_2, \tag{14}$$

with $c_{1,2} = c_{1,2}(0)$ given above.

Expressing the cross-sections directly in barn leads to

$$\sigma_{RHI} = 1.58\cdot 10^{-11} Z_t^5 Z_p^2 f(Z_t)[c_1(Z_t)\log(\gamma_p) - c_2(Z_t)]. \tag{15}$$

The formula above describes the equivalent photon cross-section with an accuracy better than 2% for $Z_t \leqslant 92$ and typical RHIC and LHC energies. It should be mentioned that it has already been suggested in [14] that the cross-section σ_{RHI} can be established in the form

$$\sigma_{RHI} = A\log\gamma_p + B. \tag{16}$$

The coefficients A and B were obtained from numerical calculations. In our case, the coefficients follow directly from eqs. (13)–(15).

Cross-sections for several nuclei and energies have been calculated, *e.g.*, by Meier *et al.* [9] and Baltz *et al.* [15]. For a RHIC Cu-Cu collision with 100 GeV/nucleon, one obtains with $\gamma_c \simeq 107$ and $\gamma_p = 2\gamma_c^2 - 1$ a cross-section of 0.198 barn, in agreement with recent experimental measurements [4]. In the case of the LHC, one obtains from eq. (15) for a Pb-Pb collision with $\gamma_c = 2957$ a cross-section of 220 barn for K-shell capture, compared to 225 barn in [9]. For $Z_p = Z_t = 20$ and $\gamma_c = 125$ ($\gamma_c = 3750$), the cross-section for K-shell capture obtained from the equivalent photon method is $1.57\cdot 10^{-2}$ barn ($2.96\cdot 10^{-2}$ barn). The corresponding values found in [9] are $1.61\cdot 10^{-2}$ barn and $2.92\cdot 10^{-2}$ barn, respectively. A cross-section of 89 barn for Au-Au collisions at RHIC energy given in [15] is also in excellent agreement with eq. (15). This demonstrates the validity of the equivalent photon method for the full physical range of charge numbers Z and large γ_c.

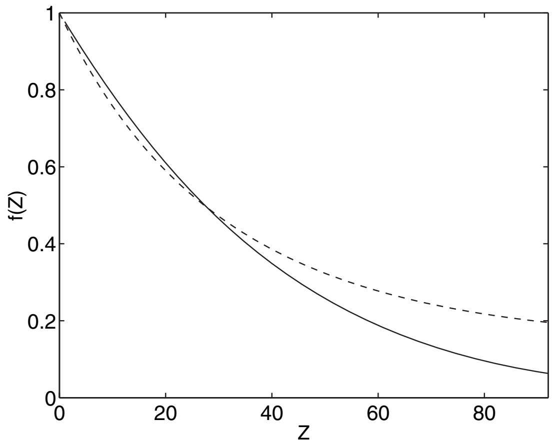

Fig. 2: Comparison of the Z-dependence of $\sigma^K_{e^+e^-}$, expressed by $\tilde{f}(z)$ from eq. (18) (solid line), and exact values for $f(Z)$ (dashed line).

As an interesting final remark, we briefly compare the results obtained above to approaches presented in [7] and in [16]. Based on Sommerfeld-Maue wave functions for positrons and semi-relativistic electron wave functions constructed from the non-relativistic electron wave functions by including perturbative correction terms of the order αZ, an approximate expression for pair production with K-shell capture was derived in [7] ($Z = Z_t = Z_p$)

$$\sigma^K_{RHI} = \frac{33}{20}\lambda_c^2\alpha^7 Z^7 \frac{2\pi\alpha Z}{e^{2\pi\alpha Z}-1}\Big[\log(\delta\gamma_p/2) - 5/3\Big]. \tag{17}$$

The numerical factor $33/20 = 1.65$ agrees in a satisfactory manner with the corresponding factor $8c_1 \cong 1.74$ appearing in eq. (11). The deviation of the cross-section from the Z^5-scaling is basically described by

$$\tilde{f}(Z) = \frac{2\pi\alpha Z}{e^{2\pi\alpha Z}-1}. \tag{18}$$

A comparison of $\tilde{f}(z)$ with $f(Z)$ obtained from exact numerical calculations is presented in fig. 2. Whereas

$\tilde{f}(Z)$ is very acceptable for $\alpha Z \ll 1$, it leads to an underestimation of cross-sections for heavy nuclei by a factor of about 2. The approach presented in [16] uses the same bound electron wave functions as in [7], however, positron wave functions are described by normalized plane waves in conjunction with a correction term proportional to αZ. The approximation in [16] suffers from the opposite effect, *i.e.*, cross-sections for nuclei with large charge numbers turn out to be too large by factors exceeding 2. This observation shows that is it crucial to work with exact solutions of the Dirac equation in order to get reliable results in the present case.

Conclusions. – A heuristic analytic fit for the characteristic function $f(Z)$ describing the single-photon pair production with capture on fully stripped nuclei at high energies is presented in eq. (10). This fit allows to describe the single-photon pair production cross-section at high energies with an accuracy comparable to presently available numerical calculations. Furthermore, eq. (15) in conjunction with eqs. (10), (13), (14) allows to evaluate the bound-free pair production cross-section in relativistic heavy-ion collisions and the corresponding coefficients A and B in eq. (16) in a simple and reliable manner. It should be mentioned that in currently available calculations, multi-photon interactions between the photon-emitting nuclei are neglected, which might lead to an additional correction of the cross-section of a few percent, probably with different energy dependence from the overall cross-section.

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