# Perturbative gauge invariance: electroweak theory II

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**Abstract.** A recent construction of the electroweak theory, based on perturbative quantum gauge invariance alone, is extended to the case of more generations of fermions with arbitrary mixing. The conditions implied by second order gauge invariance lead to an isolated solution for the fermionic couplings in agreement with the standard model. Third order gauge invariance determines the Higgs potential. The resulting massive gauge theory is manifestly gauge invariant, after construction.

## 1 Introduction

In a recent paper [1] we have described a formulation of the electroweak theory in terms of asymptotic fields alone. Since in our causal approach the asymptotic gauge fields are massive from the very beginning (except the photon), spontaneous symmetry breaking plays no role.

Let us summarize the strategy. The fundamental problem in gauge theories is the appearance of *unphysical fields* in the gauge potentials. An additional main difficulty is that already the quantization of the massive[3] free gauge fields in a $\lambda$-gauge requires an indefinite metric space. To solve these two problems in the *free* theory one has to select the space of physical states $\mathcal{H}_{\text{phys}}$ which must be a (pre) Hilbert space, i.e. the inner product must be positive definite. There are two popular methods at hand: the Gupta Bleuler method (which works well for abelian gauge theories only) and the BRST-formalism [2]. We take the latter in the form of Kugo and Ojima [3]: the physical (pre) Hilbert space is defined as the cohomology of a nilpotent operator $Q$, i.e. $\mathcal{H}_{\text{phys}} = \frac{\text{Ker}\, Q}{\text{Ran}\, Q}$. To define such an operator we introduce unphysical fields: each gauge field $A_a^\mu$ gets three scalar partners, the

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[3] This problem appears also for the massless free gauge fields. The peculiarity of the massive (free) theory is that we need the unphysical bosonic scalars $\Phi_a$ to restore the nilpotency of $Q$.

fermionic fields $u_a$, $\tilde{u}_a$ ("ghost fields") and the bosonic $\Phi_a$. Then

$$Q \overset{\text{def}}{=} \int \mathrm{d}^3x \, (\partial_\nu A_a^\nu + m_a \Phi_a) \, \overset{\leftrightarrow}{\partial}_0 u_a \, ,$$

does the job.

To overcome the presence of unphysical fields in the *interacting theory* one has additionally to prove that they *decouple*. The latter means that in the adiabatic limit (if it exists) the *S*-matrix induces a well-defined unitary operator. For this purpose one needs a notion of gauge invariance. The most natural formulation which yields the decoupling is to require that $Q$ commutes with the *S*-matrix in the adiabatic limit (if it exists). This is essentially the content of our formulation of gauge invariance

$$d_Q T_n \overset{\text{def}}{=} [Q, \, T_n] = i \sum_{l=1}^n \frac{\partial}{\partial x_l^\nu} \, T_{n/l}^\nu \, , \tag{1.0}$$

(where we use the notations of [1]), which is independent of the adiabatic limit, i.e. it makes sense also in theories in which this limit does not exist. This is a pure quantum formulation of gauge invariance. It has turned out that this symmetry requirement (1.0) cannot be satisfied with the fields at hand so far. By introducing an additional scalar field (the "Higgs field"), which is physical, gauge invariance can be saved. In addition (1.0) is sufficiently strong to determine the couplings of all fields.

The main steps in this construction are the following. Starting from the pure Yang-Mills coupling $T_1^A$ at first order, gauge invariance requires couplings $T_1^u$ of the ghost fields $u_a \tilde{u}_a$ and fixes them. This step is the same as in the massless theory [4]. However, the massive gauge fields need couplings to the bosonic scalar partners $\Phi_a$ to compensate mass terms in the first order gauge variation. Gauge invariance then determines the couplings $T_1^\Phi$ of these unphysical scalars to the gauge fields and to the ghosts. But the resulting theory would not be gauge invariant at second order. This requires the "Higgs field" $\Phi_0$, its coupling $T_1^{\Phi_0}$ to all other fields is again determined by gauge invariance. The coupling to leptons is uniquely fixed by first and second order gauge invariance if one family of fermions is considered [1]. However, it is well known that gauge invariance is then violated at third order by the triangular anomalies. Hence, the theory must still be enlarged by including at least one complete generation of leptons and quarks. In what follows we consider the general situation of arbitrarily many generations of fermions with arbitrary mixing. It is our aim to analyse what gauge invariance has to say about the couplings in this case.

Assuming that the reader is familiar with the main results of [1], we present in the following section the general ansatz for the fermionic coupling $T_1^F$ and determine the constraints imposed by gauge invariance at first order. At second order (Sect. 3) we obtain a long list of interesting conditions for the coupling matrices. At present we cannot control all solutions of this system. But inserting a general ansatz which is valid in the vicinity of the standard model, we get a unique solution that agrees with the standard model. In Sect. 4 we briefly discuss gauge invariance of the third order triangular graphs. We recover the well-known fact that the cancellation of the axial anomalies restricts the charges of the fermions.

In Sect. 5 we show how gauge invariance of third order tree graphs determines the last free parameters in the scalar coupling. In this way we *derive* the Goldstone-Higgs potential. Other approaches where this quartic double-well potential is not postulated but deduced, are based on methods of non-commutative geometry which have been developed by Connes [5] and others [6, 7, 8, 9, 10], or on super-connections [11]. The scalar potential comes out in the form

$$V(\Phi_0) = \lambda(\Phi_0^4 - 4a\Phi_0^3 + 4a^2\Phi_0^2), \tag{1.1}$$

with

$$a = \frac{2}{g}\, m_W, \tag{1.2}$$

where $m_W$ is the $W$-mass and $g$ the universal gauge coupling constant. The physical 'Higgs' field $\Phi_0$ is realized in an ordinary Fock representation with unique vacuum and vacuum expectation value $\langle\Phi_0\rangle = 0$. (In the usual convention $-\Phi_0$ is used instead of $\Phi_0$.) If we introduce the shifted field

$$\varphi = \Phi_0 - a, \tag{1.3}$$

the potential assumes the usual symmetric form [12]

$$V = \lambda(\varphi^2 - a^2)^2. \tag{1.4}$$

Now the vacuum expectation value $\langle\varphi\rangle = -a$ is different from 0. If the shift (1.3) is carried out everywhere in the scalar couplings (see (3.20) of [1]) we get quadratic mass terms (with the wrong sign) for the already massive gauge bosons. These quadratic terms can simply be resummed which changes the $W$- and $Z$-masses from their finite values into zero! That means, we have inverted the Higgs mechanism. This gives the connection of our approach with the standard theory. However, this connection does not enter our construction.

## 2 Fermionic coupling and first order gauge invariance

We start from the following generalization of the simple leptonic electroweak coupling to more than one family

$$T_1^F = ig \sum_{j,k} \big\{ W_\mu^+ b_{jk}^1 \bar{e}_j \gamma^\mu \nu_k + W_\mu^+ b_{jk}'^1 \bar{e}_j \gamma^\mu \gamma^5 \nu_k \tag{2.1.1}$$

$$+ W_\mu^- b_{jk}^2 \bar{\nu}_j \gamma^\mu e_k + W_\mu^- b_{jk}'^2 \bar{\nu}_j \gamma^\mu \gamma^5 e_k \tag{2.1.2}$$

$$+ Z_\mu b_{jk}^3 \bar{e}_j \gamma^\mu e_k + Z_\mu b_{jk}'^3 \bar{e}_j \gamma^\mu \gamma^5 e_k \tag{2.1.3}$$

$$+ Z_\mu b_{jk}^4 \bar{\nu}_j \gamma^\mu \nu_k + Z_\mu b_{jk}'^4 \bar{\nu}_j \gamma^\mu \gamma^5 \nu_k \tag{2.1.4}$$

$$+ A_\mu b_{jk}^5 \bar{e}_j \gamma^\mu e_k + A_\mu b_{jk}'^5 \bar{e}_j \gamma^\mu \gamma^5 e_k \tag{2.1.5}$$

$$+ A_\mu b_{jk}^6 \bar{\nu}_j \gamma^\mu \nu_k + A_\mu b_{jk}'^6 \bar{\nu}_j \gamma^\mu \gamma^5 \nu_k \tag{2.1.6}$$

$$+ \Phi^+ c_{jk}^1 \bar{e}_j \nu_k + \Phi^+ c_{jk}'^1 \bar{e}_j \gamma^5 \nu_k + \Phi^- c_{jk}^2 \bar{\nu}_j e_k + \Phi^- c_{jk}'^2 \bar{\nu}_j \gamma^5 e_k \tag{2.1.7}$$

$$+ \Phi_3 c_{jk}^3 \bar{e}_j e_k + \Phi_3 c_{jk}'^3 \bar{e}_j \gamma^5 e_k + \Phi_3 c_{jk}^4 \bar{\nu}_j \nu_k + \Phi_3 c_{jk}'^4 \bar{\nu}_j \gamma^5 \nu_k \tag{2.1.8}$$

$$+ \Phi_0 c_{jk}^0 \bar{e}_j e_k + \Phi_0 c_{jk}'^0 \bar{e}_j \gamma^5 e_k + \Phi_0 c_{jk}^5 \bar{\nu}_j \nu_k + \Phi_0 c_{jk}'^5 \bar{\nu}_j \gamma^5 \nu_k \big\}. \tag{2.1.9}$$

Here we have used the same notation as in [1] (5.1): All products of field operators throughout are normally ordered (Wick monomials). $W, Z, A$ denote the gauge fields and $\Phi^{\pm} = (\Phi_1 \pm i\Phi_2)/\sqrt{2}$, $\Phi_3$ the unphysical scalars. $\Phi_0$ is the physical scalar, but $e_j(x)$ stands for the electron-, muon-, tau-fields, as well as for the quark fields $d$, $s$, $b$, and $\nu_k(x)$ represents the corresponding neutrini and the other quark fields $u$, $c$, $t$. In [1] we have only considered the coupling to leptons. There the terms (2.1.6) are missing because the neutrini have vanishing electric charge, but here, for the quark couplings, we must include them. We also assume that the asymptotic Fermi fields fulfil the Dirac equations

$$\partial\!\!\!/ e_j = -im_j^e e_j\,, \qquad \partial_\mu(\bar{e}_j\gamma^\mu) = im_j^e \bar{e}_j\,,$$
$$\partial\!\!\!/ \nu_k = -im_k^\nu \nu_k\,, \qquad \partial_\mu(\bar{\nu}_k\gamma^\mu) = im_k^\nu \bar{\nu}_k, \tag{2.2}$$

with arbitrary non-vanishing unequal masses $m_j^e \neq m_k^e \neq 0$, $m_j^\nu \neq m_k^\nu \neq 0$ for all $j \neq k$. We do not use further information about the multiplett structure of the fermions.

According to [1] the gauge structure is introduced as follows. We define a gauge charge

$$Q \stackrel{\text{def}}{=} \int \mathrm{d}^3x \sum_{a=0}^{3} \left(\partial_\nu A_a^\nu + m_a \Phi_a\right) \stackrel{\leftrightarrow}{\partial}_0 u_a\,,$$

where $A_a^\nu$ stands for $A^\nu$, $W_1^\nu$, $W_2^\nu$, $Z^\nu$ and $u_a$ for the corresponding ghosts, and a gauge variation

$$d_Q F \stackrel{\text{def}}{=} QF - (-1)^{n_F} FQ\,.$$

Then the gauge variations of the asymptotic gauge fields are given by

$$d_Q A^\mu = i\,\partial^\mu u_0\,, \qquad d_Q W_{1,2}^\mu = i\,\partial^\mu u_{1,2}\,, \qquad d_Q Z^\mu = i\,\partial^\mu u_3\,,$$

and for the Higgs and unphysical scalar fields

$$d_Q \Phi_0 = 0\,, \qquad d_Q \Phi_{1,2} = im_W u_{1,2}\,, \qquad d_Q \Phi_3 = im_Z u_3$$

and finally for the fermionic ghosts

$$d_Q u_a = 0\,, \qquad a = 0, 1, 2, 3\,,$$
$$d_Q \tilde{u}_0 = -i\,\partial_\mu A^\mu\,, \qquad d_Q \tilde{u}_{1,2} = -i(\partial_\mu W_{1,2}^\mu + m_W \Phi_{1,2})\,,$$
$$d_Q \tilde{u}_3 = -i(\partial_\mu Z^\mu + m_Z \Phi_3)\,. \tag{2.3}$$

The gauge variations of the fermionic matter fields vanish.

First order gauge invariance means that the gauge variation of

$$T_1 = T_1^A + T_1^u + T_1^\Phi + T_1^{\Phi_0} + T_1^F$$

has divergence form

$$d_Q T_1(x) = i\,\partial_\mu T_{1/1}^\mu\,. \tag{2.4}$$

To verify this for the fermionic coupling $T_1^F$ we calculate the gauge variation of (2.1) and take out the derivatives of the ghost fields. In the additional terms with

derivatives on the matter fields we use the Dirac equations (2.2):

$$d_Q T_1^F = -g \sum_{j,k} \partial_\mu \{ u^+(b_{jk}^1 \bar{e}_j \gamma^\mu \nu_k + b_{jk}'^1 \bar{e}_j \gamma^\mu \gamma^5 \nu_k) + u^-(b_{jk}^2 \bar{\nu}_j \gamma^\mu e_k + b_{jk}'^2 \bar{\nu}_j \gamma^\mu \gamma^5 e_k) \qquad (a)$$

$$+ u_3(b_{jk}^3 \bar{e}_j \gamma^\mu e_k + b_{jk}'^3 \bar{e}_j \gamma^\mu \gamma^5 e_k + b_{jk}^4 \bar{\nu}_j \gamma^\mu \nu_k + b_{jk}'^4 \bar{\nu}_j \gamma^\mu \gamma^5 \nu_k) \qquad (b)$$

$$+ u_0(b_{jk}^5 \bar{e}_j \gamma^\mu e_k + b_{jk}'^5 \bar{e}_j \gamma^\mu \gamma^5 e_k + b_{jk}^6 \bar{\nu}_j \gamma^\mu \nu_k + b_{jk}'^6 \bar{\nu}_j \gamma^\mu \gamma^5 \nu_k) \} \qquad (c)$$

$$+ g \{ i u^+ [b_{jk}^1 (m_j^e - m_k^\nu) \, \bar{e}_j \nu_k + b_{jk}'^1 (m_j^e + m_k^\nu) \, \bar{e}_j \gamma^5 \nu_k] \qquad (d)$$

$$+ i u^- [b_{jk}^2 (m_j^\nu - m_k^e) \, \bar{\nu}_j e_k + b_{jk}'^2 (m_j^\nu + m_k^e) \, \bar{\nu}_j \gamma^5 e_k]$$

$$+ i u_3 [b_{jk}^3 (m_j^e - m_k^e) \, \bar{e}_j e_k + b_{jk}'^3 (m_j^e + m_k^e) \, \bar{e}_j \gamma^5 e_k$$

$$+ b_{jk}^4 (m_j^\nu - m_k^\nu) \, \bar{\nu}_j \nu_k + b_{jk}'^4 (m_j^\nu + m_k^\nu) \, \bar{\nu}_j \gamma^5 \nu_k]$$

$$+ i u_0 [b_{jk}^5 (m_j^e - m_k^e) \, \bar{e}_j e_k + b_{jk}'^5 (m_j^e + m_k^e) \, \bar{e}_j \gamma^5 e_k$$

$$+ b_{jk}^6 (m_j^\nu - m_k^\nu) \, \bar{\nu}_j \nu_k + b_{jk}'^6 (m_j^\nu + m_k^\nu) \, \bar{\nu}_j \gamma^5 \nu_k] \}$$

$$+ m_W [u^+ c_{jk}^1 \bar{e}_j \nu_k + u^+ c_{jk}'^1 \bar{e}_j \gamma^5 \nu_k + u^- c_{jk}^2 \bar{\nu}_j e_k + u^- c_{jk}'^2 \bar{\nu}_j \gamma^5 e_k]$$

$$+ m_Z u_3 [c_{jk}^3 \bar{e}_j e_k + c_{jk}'^3 \bar{e}_j \gamma^5 e_k + c_{jk}^4 \bar{\nu}_j \nu_k + c_{jk}'^4 \bar{\nu}_j \gamma^5 \nu_k] \}. \qquad (2.5)$$

Now, to have first order gauge invariance, the terms $(d)$ until the end of (2.5) which are not of divergence form must cancel. This implies

$$b_{jk}'^5 = 0 = b_{jk}'^6, \qquad \forall j, k, \qquad b_{jk}^5 = 0 = b_{jk}^6 \quad \text{for} \quad j \neq k \qquad (2.6)$$

and

$$c_{jk}^1 = \frac{i}{m_W} (m_j^e - m_k^\nu) \, b_{jk}^1, \qquad c_{jk}'^1 = \frac{i}{m_W} (m_j^e + m_k^\nu) \, b_{jk}'^1,$$

$$c_{jk}^2 = \frac{i}{m_W} (m_j^\nu - m_k^e) \, b_{jk}^2, \qquad c_{jk}'^2 = \frac{i}{m_W} (m_j^\nu + m_k^e) \, b_{jk}'^2,$$

$$c_{jk}^3 = \frac{i}{m_Z} (m_j^e - m_k^e) \, b_{jk}^3, \qquad c_{jk}'^3 = \frac{i}{m_Z} (m_j^e + m_k^e) \, b_{jk}'^3,$$

$$c_{jk}^4 = \frac{i}{m_Z} (m_j^\nu - m_k^\nu) \, b_{jk}^4, \qquad c_{jk}'^4 = \frac{i}{m_Z} (m_j^\nu + m_k^\nu) \, b_{jk}'^4. \qquad (2.7)$$

The result (2.6) means that the photon has no axial-vector coupling and no mixing in the vector coupling. This is due to the fact that it has no scalar partner because it is massless.

## 3 Gauge Invariance at Second Order

As discussed in Sect. 4 and 5 of [1], the essential problem in second order gauge invariance is whether the anomalies in the tree graphs cancel out. These anomalies are the local terms in $\partial_\nu^x T_{2/1}^\nu|_{\text{tree}}^0 (x, y) + \partial_\nu^y T_{2/2}^\nu|_{\text{tree}}^0 (x, y)$ and come from two sources. First, if the terms $(a) - (c)$ in (2.5) are combined with the terms in (2.1) by a

fermionic contraction we get the Feynman propagator $S_m^F(x - y)$ in $T_{2/1}^\nu\big|_{\text{tree}}^0(x, y)$ and $T_{2/2}^\nu\big|_{\text{tree}}^0(x, y)$. Taking the divergence with respect to the $Q$-vertex a $\delta$-distribution is generated due to

$$i\, \partial_\mu^x \gamma^\mu \, S_m^F(x - y) = m S_m^F(x - y) + \delta(x - y). \tag{3.1}$$

This $\delta$-term is the anomaly. Secondly, we can perform a bosonic contraction between the terms in (2.1) and the terms (4.5) in [1] which are the anomaly-producing part in $T_{1/1}^{A\Phi\mu}$, coming from the Yang-Mills and scalar couplings:

$$T_{1/1}^{A\Phi\mu}\big|_{\text{an}} = ig\left\{\sin\Theta(u_1 W_2^\nu - u_2 W_1^\nu)\,\partial^\mu A_\nu\right. \tag{3.2.1}$$

$$+\sin\Theta(u_2 A^\nu - u_0 W_2^\nu)\,\partial^\mu W_{1\nu} + \sin\Theta(u_0 W_1^\nu - u_1 A^\nu)\,\partial^\mu W_{2\nu} \tag{3.2.2}$$

$$+\cos\Theta\big[(u_2 Z^\nu - u_3 W_2^\nu)\,\partial^\mu W_{1\nu} + (u_3 W_1^\nu - u_1 Z^\nu)\,\partial^\mu W_{2\nu} \tag{3.2.3}$$

$$+(u_1 W_2^\nu - u_2 W_1^\nu)\,\partial^\mu Z_\nu\big] \tag{3.2.4}$$

$$+\sin\Theta(u_0 u_1\,\partial^\mu \tilde{u}_2 + u_2 u_0\,\partial^\mu \tilde{u}_1 + u_1 u_2\,\partial^\mu \tilde{u}_0) \tag{3.2.5}$$

$$+\cos\Theta(u_2 u_3\partial^\mu \tilde{u}_1 + u_3 u_1\partial^\mu \tilde{u}_2 + u_1 u_2\partial^\mu \tilde{u}_3) \tag{3.2.6}$$

$$+\sin\Theta u_0(\Phi_2\partial^\mu \Phi_1 - \Phi_1\partial^\mu \Phi_2) + \left(1 - \frac{m_Z^2}{2m_W^2}\right)\cos\Theta u_3(\Phi_2\partial^\mu \Phi_1 - \Phi_1\partial^\mu \Phi_2) \tag{3.2.7}$$

$$+\frac{1}{2}\frac{m_Z}{m_W}\cos\Theta\big[(u_2\Phi_1 - u_1\Phi_2)\,\partial^\mu \Phi_3 + u_1\Phi_3\partial^\mu \Phi_2 - u_2\Phi_3\partial^\mu \Phi_1\big] \tag{3.2.8}$$

$$+\frac{1}{2}u_1(\Phi_0\partial^\mu \Phi_1 - \Phi_1\partial^\mu \Phi_0) + \frac{1}{2}u_2(\Phi_0\partial^\mu \Phi_2 - \Phi_2\partial^\mu \Phi_0) \tag{3.2.9}$$

$$+\frac{1}{2\cos\Theta}u_3(\Phi_0\partial^\mu \Phi_3 - \Phi_3\partial^\mu \Phi_0)\bigg\}. \tag{3.2.10}$$

To give a representative example, we calculate the anomalies with external field operators $u_3\Phi_3\bar{e}_j\gamma^5 e_k$. Combining the first term in (2.5) (b) with the second term in (2.1.8) we get an anomaly from the contraction of $e_{k'}(x)$ with $\bar{e}_{j'}(y)$ ($S^F[e_{k'}(x), \bar{e}_{j'}(y)]$ denotes the corresponding Feynman propagator)

$$-i u_3(x)\, b_{jk'}^3\, \bar{e}_j(x)\, \gamma^\mu S^F[e_{k'}(x),\, \bar{e}_{j'}(y)]\, \gamma^5 e_k(y)\, \Phi_3(y)\, c_{j'k}'^3.$$

It results the anomaly $\sim -i b^3 c'^3 \delta(x - y)$ involving the matrix product of $b^3$ with $c'^3$. Combining the two terms with reversed order, we obtain $c'^3 b^3$ with a different sign so that both terms together yield the commutator $i[c'^3, b^3]$. Similarly, the second term in (2.5) (b) together with the first term in (2.1.8) gives the anticommutator $\{c^3, b'^3\}$, because the $\gamma^5$ is at a different place. An anomaly of the second source comes from the last term in (3.2.10) contracted by the two $\Phi_0$-fields with the second term in (2.1.9):

$$-\frac{i}{2\cos\Theta}u_3(x)\,\Phi_3(x)\,D^F[\partial^\mu \Phi_0(x),\,\Phi_0(y)]\,\bar{e}_j(y)\,\gamma^5 e_k(y)\,c_{jk}'^0.$$

Altogether we obtain the following matrix equation

$$-\frac{1}{2\cos\Theta}\, c'^0 = i[c'^3,\, b^3] + i\{c^3,\, b'^3\}\,. \tag{3.3}$$

We now give the complete list of all second order conditions. We specify the corresponding external legs, then the origin of the terms is pretty clear. Every combination of external field operators has a corresponding one with an additional $\gamma^5$. To save space we do not write down the external legs once more for the $\gamma^5$-term.

$u_0\Phi_0\bar{e}e :\ [c^0,\, b^5] = 0\,,\qquad [c'^0,\, b^5] = 0$

$u_0\Phi_3\bar{e}e :\ [c^3,\, b^5] = 0\,,\qquad [c'^3,\, b^5] = 0$

$u_0 Z\bar{e}\gamma e :\ [b^3,\, b^5] = 0\,,\qquad [b'^3,\, b^5] = 0$

$u_0\Phi_0\bar{\nu}\nu :\ [c^5,\, b^6] = 0\,,\qquad [c'^5,\, b^6] = 0$

$u_0\Phi_3\bar{\nu}\nu :\ [c^4,\, b^6] = 0\,,\qquad [c'^4,\, b^6] = 0$

$u_0 Z\bar{\nu}\gamma\nu :\ [b^4,\, b^6] = 0\,,\qquad [b'^4,\, b^6] = 0 \tag{3.4}$

$u^+ A\bar{e}\gamma\nu :\ \sin\Theta b^1 = b^5 b^1 - b^1 b^6\,,\qquad \sin\Theta b'^1 = b^5 b'^1 - b'^1 b^6$

$u^- A\bar{\nu}\gamma e :\ \sin\Theta b^2 = b^2 b^5 - b^6 b^2\,,\qquad \sin\Theta b'^2 = b'^2 b^5 - b^6 b'^2 \tag{3.5}$

$u_0\Phi^+\bar{e}\nu :\ \sin\Theta c^1 = b^5 c^1 - c^1 b^6\,,\qquad \sin\Theta c'^1 = b^5 c'^1 - c'^1 b^6$

$u_0\Phi^-\bar{\nu}e :\ \sin\Theta c^2 = c^2 b^5 - b^6 c^2\,,\qquad \sin\Theta c'^2 = c'^2 b^5 - b^6 c'^2 \tag{3.6}$

$u_3 W^+\bar{e}\gamma\nu :\ \cos\Theta b^1 = b^3 b^1 + b'^3 b'^1 - b^1 b^4 - b'^1 b'^4\,,$
$\qquad\qquad \cos\Theta b'^1 = b'^3 b^1 + b^3 b'^1 - b'^1 b^4 - b^1 b'^4$

$u_3 W^-\bar{\nu}\gamma e :\ \cos\Theta b^2 = b^2 b^3 + b'^2 b'^3 - b^4 b^2 - b'^4 b'^2\,,$
$\qquad\qquad \cos\Theta b'^2 = b'^2 b^3 + b^2 b'^3 - b'^4 b^2 - b^4 b'^2 \tag{3.7}$

$u^+ W^-\bar{e}\gamma e :\ \cos\Theta b^3 = b^1 b^2 + b'^1 b'^2 - \sin\Theta b^5\,,\qquad \cos\Theta b'^3 = b'^1 b^2 + b^1 b'^2$

$u^- W^+\bar{\nu}\gamma\nu :\ \cos\Theta b^4 = -b^2 b^1 - b'^2 b'^1 - \sin\Theta b^6\,,\qquad \cos\Theta b'^4 = -b'^2 b^1 - b^2 b'^1 \tag{3.8}$

$u+\Phi_3\bar{e}\nu :\ c^1 = 2(b'^1 c'^4 + c'^3 b'^1 + c^3 b^1 - b^1 c^4)\,,\qquad c'^1 = 2(b'^1 c^4 + c^3 b'^1 + c'^3 b^1 - b^1 c'^4)$

$u^-\Phi_3\bar{\nu}e :\ c^2 = 2(-b'^2 c'^3 - c'^4 b'^2 - c^4 b^2 + b^2 c^3)\,,$
$\qquad\qquad c'^2 = 2(-b'^2 c^3 - c^4 b'^2 - c'^4 b^2 + b^2 c'^3)\,, \tag{3.9}$

$u^+\Phi_0\bar{e}\nu :\ \delta_1 c^1 = i(c^0 b^1 + c'^0 b'^1 + b'^1 c^5 - b^1 c^5)\,,$
$\qquad\qquad \delta_1 c'^1 = i(c'^0 b^1 + c^0 b'^1 + b'^1 c^5 - b^1 c'^5)\,,$

$u^-\Phi_0\bar{\nu}e :\ \delta_1 c^2 = i(c^5 b^2 + c'^5 b'^2 + b'^2 c'^0 - b^2 c^0)\,,$
$\qquad\qquad \delta_1 c'^2 = i(c'^5 b^2 + c^5 b'^2 + b'^2 c^0 - b^2 c'^0)\,, \tag{3.10}$

$u_3\Phi_3\bar{e}e :\ \delta_3 c^0 = i[b^3,\, c^3] - i\{b'^3,\, c'^3\}\,,\qquad \delta_3 c'^0 = i[b^3,\, c'^3] - i\{b'^3,\, c^3\}$

$u_3\Phi_0\bar{e}e :\ \delta_3 c^3 = i[c^0,\, b^3] + i\{c'^0,\, b'^3\}\,,\qquad \delta_3 c'^3 = i[c'^0,\, b^3] + i\{c^0,\, b'^3\}\,, \tag{3.11}$

$u_3\Phi_3\bar{\nu}\nu :\ \delta_3 c^5 = i[b^4,\, c^4] - i\{b'^4,\, c'^4\}\,,\qquad \delta_3 c'^5 = i[b^4,\, c'^4] - i\{b'^4,\, c^4\}$

$u_3\Phi_0\bar{\nu}\nu :\ \delta_3 c^4 = i[c^5,\, b^4] + i\{c'^5,\, b'^4\}\,,\qquad \delta_3 c'^4 = i[c'^5,\, b^4] + i\{c^5,\, b'^4\}\,, \tag{3.12}$

$u^+\Phi^-\bar{e}e :\ \delta_1 c^0 = i(b^1 c^2 - b'^1 c'^2) - ic^3/2\,,\qquad \delta_1 c'^0 = i(b^1 c'^2 - b'^1 c^2) - ic'^3/2$

$u^-\Phi^+\bar{e}e :\ \delta_1 c^0 = -i(c^1 b^2 + c'^1 b'^2) + ic^3/2\,,\qquad \delta_1 c'^0 = -i(c'^1 b^2 + c^1 b'^2) + ic'^3/2\,, \tag{3.13}$

$u^+ \Phi^- \bar{\nu} \nu :\ \delta_1 c^5 = i(b^2 c^1 - b'^2 c'^1) - i c^4 / 2 \,, \qquad \delta_1 c'^5 = i(b^2 c'^1 - b'^2 c^1) - i c'^4 / 2$

$u^- \Phi^+ \bar{\nu} \nu :\ \delta_1 c^5 = -i(c^2 b^1 + c'^2 b'^1) + i c^4 / 2 \,, \quad \delta_1 c'^5 = -i(c'^2 b^1 + c^2 b'^1) + i c'^4 / 2 \ \ (3.14)$

$u_3 \Phi^+ \bar{e} \nu :\ \delta_4 c^1 = b^3 c^1 - b'^3 c'^1 - c^1 b^4 - c'^1 b'^4 \,, \qquad \delta_4 c'^1 = -b'^3 c^1 + b^3 c'^1 - c^1 b'^4 - c'^1 b^4$

$u_3 \Phi^- \bar{\nu} e :\ \delta_4 c^2 = -b^4 c^2 + b'^4 c'^2 + c^2 b^3 + c'^2 b'^3 \,, \qquad \delta_4 c'^2 = b'^4 c^2 - b^4 c'^2 + c^2 b'^3 + c'^2 b^3 \,,$

$$(3.15)$$

where

$$\delta_1 = \frac{1}{2} \,, \qquad \delta_3 = \frac{1}{2 \cos \Theta} \,, \qquad \delta_4 = \cos \Theta - \frac{1}{2 \cos \Theta} \,.$$

The terms with these $\delta$'s and with the electroweak mixing angle obviously come from (3.2). There are further combinations of external field operators which have not been written down, because they give no new condition.

In case of one family, assuming $b^6 = 0$ and taking pseudounitarity into account ([1] (5.21)), the corresponding system of scalar equations has a unique solution, which agrees with the lepton coupling of the standard model. The solution of the above matrix equations $(3.4 - 15)$ is not so simple. We start from the equations (3.5). If we write these equations with matrix elements, using the fact that $b^5$ and $b^6$ are diagonal (2.6), we easily conclude that $b^5$ and $b^6$ are actually multiples of the unit matrix

$$b^5 = \alpha \mathbf{1} \,, \qquad b^6 = (\alpha - \sin \Theta) \mathbf{1} \,. \tag{3.16}$$

Here $\alpha$ is a free parameter (the electric charge of the upper quarks or leptons) and we have assumed that the matrices $b^1, b'^1, b^2, b'^2$ are nontrivial and non-diagonal. This is not a serious limitation because we shall see that these matrices are essentially the unitary mixing matrices. For the leptons we assume $b_6 = 0$ instead. This first consequence of gauge invariance is the universality of the electromagnetic coupling: the members $e_k(x)$ and $\nu_k(x)$ of different generations all couple in the same way to the photon, with a constant charge difference $q_e - q_\nu = g \sin \Theta$, which is the electronic charge.

Next we turn to the conditions (3.11). It is convenient to introduce the diagonal mass matrices

$$m^e = \text{diag} \left( m_j^e \right) \,, \qquad m^\nu = \text{diag} \left( m_j^\nu \right) \,, \quad j = 1, \dots n_g \,,$$

where $n_g$ is the number of generations. Then (2.7) can be written as follows

$$c^1 = \frac{i}{m_W} \left( m^e b^1 - b^1 m^\nu \right) \,, \qquad c'^1 = \frac{i}{m_W} \left( m^e b'^1 + b'^1 m^\nu \right) \,, \tag{3.17}$$

etc. Then the first two equations in (3.11) read

$$c'^0 = \frac{1}{\delta_3 m_Z} \left( 2 b'^3 m^e b^3 - 2 b^3 m^e b'^3 + m^e b'^3 b^3 + m^e b^3 b'^3 - b^3 b'^3 m^e - b'^3 b^3 m^e \right) \,,$$

$$c^0 = \frac{1}{\delta_3 m_Z} \left( 2 b'^3 m^e b^3 - 2 b^3 m^e b'^3 + (b^3)^2 m^e + m^e (b^3)^2 + (b'^3)^2 m^e + m^e (b'^3)^2 \right) \,.$$

$$(3.18)$$

Substituting this into the last two equations of (3.11), we arrive at the following coupled matrix equations for $b^3$, $b'^3$:

$$\delta_3^2(m^e b'^3 + b'^3 m^e) = 3b'^3 m^e (b^3)^2 + 3(b^3)^2 m^e b'^3 + 3(b'^3)^2 m^e b'^3 + 3b'^3 m^e (b'^3)^2$$
$$- 3b^3 m^e b'^3 b^3 - 3b^3 b'^3 m^e b^3 - 3b^3 b^3 m^e b'^3 - 3b^3 m^e b^3 b'^3$$
$$+ m^e b'^3 (b^3)^2 + m^e b^3 b'^3 b^3 + m^e (b^3)^3 + m^e (b^3)^2 b'^3$$
$$+ (b^3)^2 b'^3 m^e + b^3 b'^3 b^3 m^e + b'^3 (b^3)^2 m^e + (b'^3)^3 m^e, \tag{3.19}$$

$$\delta_3^2(m^e b^3 - b^3 m^e) = 3(b'^3)^2 m^e b^3 + 3(b^3)^2 m^e b^3 + 3b'^3 m^e b'^3 b^3 + 3b'^3 m^e b^3 b'^3$$
$$- 3b^3 m^e (b^3)^2 - 3b^3 b'^3 m^e b'^3 - 3b'^3 b^3 m^e b'^3 - 3b^3 m^e (b'^3)^2$$
$$+ m^e (b^3)^3 - (b^3)^3 m^e + m^e (b'^3)^2 b^3 - (b'^3)^2 b^3 m^e$$
$$+ m^e b^3 (b'^3)^2 - b^3 (b'^3)^2 m^e + m^e b'^3 b^3 b'^3 - b'^3 b^3 b'^3 m^e. \tag{3.20}$$

The coupled cubic equations (3.19−20) have many solutions in general. To determine the solutions in the neighbourhood of the standard model, we substitute

$$b^3 = \delta_3(\beta \mathbf{1} + x), \qquad b'^3 = \delta_3(\beta' \mathbf{1} + y), \qquad \beta, \beta' \in \mathbf{C}, \tag{3.21}$$

and assume the matrices $x$, $y$ to be small so that only terms linear in $x$ and $y$ must be taken with in (3.19−20). Then the equations collapse to the simple form

$$2\beta'(1 - 4\beta'^2) m + (1 - 12\beta'^2)(my + ym) = 0, \tag{3.22}$$

$$(1 - 12\beta'^2)(mx - xm) = 0. \tag{3.23}$$

Now, (3.22) yields a unique solution if $\beta' = O(1)$ is assumed

$$\beta' = \frac{\varepsilon_2}{2}, \qquad \varepsilon_2 = \pm 1, \qquad y = 0, \tag{3.24}$$

where the last result follows by writing the vanishing anticommutator $\{m^e, y\}$ with matrix elements, using $m_j^e > 0$, $\forall j$. Then (3.23) implies

$$(m_i^e - m_k^e) x_{ik} = 0, \qquad \text{no sum over } i, k \tag{3.25}$$

that means $x$ and $b^3$ are diagonal, taking into account that the masses $m_j^e$ are not degenerate. All matrices in (3.18) commute, thus

$$c'^0 = 0, \tag{3.26}$$

$$c^0 = \frac{4m^e}{\delta_3 m_Z} (b'^3)^2 = \frac{\delta_3}{m_Z} m^e, \tag{3.27}$$

and (3.11) gives

$$c_3 = 0, \qquad c'^3 = i\varepsilon_2 \frac{\delta_3}{m_Z} m^e. \tag{3.28}$$

The same reasoning can be carried through for (3.12) which leads to

$$b'^4 = -\varepsilon_2 \frac{\delta_3}{2}, \qquad c^4 = 0, \tag{3.29}$$

for the sign see below, and

$$c'^4 = -i\varepsilon_2 \frac{\delta_3}{m_Z} m^\nu, \qquad c'^5 = 0, \tag{3.30}$$

$$c^5 = \frac{\delta_3}{m_Z} m^\nu. \tag{3.31}$$

With this knowledge we turn to (3.9). Substituting $c^1$ in the first equation by (3.17) we arrive at

$$m^e(b^1 - \varepsilon_2 b'^1) = (b^1 - \varepsilon_2 b'^1) m^\nu,$$

leading to

$$b'^1 = \varepsilon_2 b^1, \tag{3.32}$$

assuming non-degenerate masses again. In the same way the second equation in (3.9) yields

$$b'^2 = \varepsilon_2 b^2. \tag{3.33}$$

This is the chiral coupling of all fermion generations. The sign in (3.29) follows from (3.8).

Finally, from the four conditions (3.8) it is easy to conclude

$$b^2 = \tfrac{1}{8} (b^1)^{-1}, \tag{3.34}$$

$$b^3 = \frac{1}{\cos\Theta} \left( \frac{1}{4} - \alpha \sin\Theta \right), \tag{3.35}$$

$$b^4 = \frac{1}{\cos\Theta} \left( -\frac{1}{4} - \alpha \sin\Theta + \sin^2\Theta \right). \tag{3.36}$$

This means that $x$ in (3.21) is actually zero, so that *there is no other solution in the neighbourhood of the standard model*. But solutions "far away" are not excluded. All values of the $b$'s and $c$'s agree with the standard model [12] for an arbitrary number $n_g$ of generations. It is not hard to check that with the results so determined all other second order conditions of gauge invariance are satisfied. To finish this discussion we notice that pseudo-unitarity implies

$$b^{1+} = b^2 = \tfrac{1}{8} (b^1)^{-1},$$

hence

$$b^1 = \frac{1}{2\sqrt{2}} V, \qquad b^2 = \frac{1}{2\sqrt{2}} V^+, \tag{3.37}$$

where $V$ is an arbitrary unitary matrix. This is the CKM mixing matrix [12, 13] for the quark coupling. A similar mixing is possible for the charged leptonic currents. The recently observed signals of neutrino oscillations show that this mixing probably occurs.

## 4 Gauge invariance at third order: axial anomalies

Adler, Bell and Jackiw [14] discovered that there exists a possibility to violate gauge invariance at third order in the triangular graphs. This holds also true in

the causal approach to gauge theory ([15] sect. 5.3). The anomalous graphs contain one axial-vector and two vector couplings (VVA) or three axial-vector couplings (AAA) of the fermions and three external gauge fields. To have a more compact notation, we collect all fermionic matter fields into a big vector $\psi = (e, \mu, \tau, \nu_e, \nu_\mu, \nu_\tau)$ or $= (d, s, b, u, c, t)$, respectively. The gauge fields $A$, $W^+$, $W^-$, $Z$ are denoted by $A_a^\mu$ with $a = 0, +, -, 3$. Then the coupling between fermions and gauge fields in (2.1) can be written as

$$T_1^{FA} = ig\{\bar\psi\gamma_\mu M_a\psi A_a^\mu + \bar\psi\gamma_\mu\gamma^5 M_a'\psi A_a^\mu\},\tag{4.1}$$

where $M_a$ stands for the following matrices of matrices:

$$M_+ = \begin{pmatrix} 0 & b^1 \\ 0 & 0 \end{pmatrix},\qquad M_- = \begin{pmatrix} 0 & 0 \\ b^2 & 0 \end{pmatrix},\tag{4.2}$$

$$M_3 = \begin{pmatrix} b^3 & 0 \\ 0 & b^4 \end{pmatrix},\qquad M_0 = \begin{pmatrix} b^5 & 0 \\ 0 & b^6 \end{pmatrix},\tag{4.3}$$

and similarly for the axial-vector couplings, denoted by a prime.

Each triangular graph gives rise to two diagrams which differ by a permutation of two vertices. Therefore, we have to compute the traces

$$\mathrm{tr}\,(M_a M_b M_c) + \mathrm{tr}\,(M_a M_c M_b) = \mathrm{tr}\,(M_a\{M_b, M_c\}),$$

where one or three $M$'s must be axial-vector couplings with a prime. It is well-known that the cancellation of the axial anomalies relies on the compensation of these traces in the sum of the leptonic and hadronic contributions. In this way third order gauge invariance gives a further restriction of the quark coupling. To work this out in detail, we consider the following cases.

A. Case $(a = 0,\ b = +,\ c = -)_{VVA}$:

$$\mathrm{tr}\,(M_0\{M_+', M_-\}) + \mathrm{tr}\,(M_0\{M_+, M_-'\})$$
$$= \mathrm{tr}\,[b^5(b^1 b'^2 + b'^1 b^2) + b^6(b^2 b'^1 + b'^2 b^1)].$$

Using the results of the last section, this is equal to

$$= n_g\,\frac{\varepsilon_2}{4}\,(2\alpha - \sin\Theta),\tag{4.4}$$

where $n_g$ is the number of generations, i.e. the dimension of the matrices $b^k$, and $\alpha$ is the charge of the fermions in (3.16). For leptons we have $b^6 = 0$, because the neutrini have no electric charge, so that

$$\alpha_L = \sin\Theta.\tag{4.5}$$

Consequently, to compensate the triangular anomaly proportional to (4.4), one needs the compensation between leptons and quarks. We assume equal number of families in the lepton and quark sectors. Then for three colors of quarks one must have

$$2\alpha_L - \sin\Theta + 3(2\alpha_Q - \sin\Theta) = 0,\tag{4.6}$$

which implies

$$\alpha_Q = \tfrac{1}{3} \sin \Theta \tag{4.7}$$

by (4.5).

In most textbooks the electric charge $\alpha_Q$ (4.7) of the $d$-, $s$-, $b$-quarks is put in and then, by requiring cancellation of the anomalies, one concludes that the number of families in the lepton and quark sectors must be equal. We have simply reversed the argument.

*B. Case* $(0, 3, 3)_{VVA}$:
In this case the trace is simply given by

$$\text{tr}\,(M_0\{M_3', M_3\}) = \text{tr}\,[b^5(b'^3 b^3 + b^3 b'^3) + b^6(b'^4 b^4 + b^4 b'^4)]$$
$$= n_g \varepsilon_2 \frac{\delta_3}{\cos \Theta} \left(\frac{1}{4} - \sin^2 \Theta\right)(2\alpha - \sin \Theta). \tag{4.8}$$

Due to the same factor $(2\alpha - \sin \Theta)$ as in (4.4) the mechanism of compensation is the same.

*C. Case* $(3, +, -)_{VVA}$:
Here the trace is equal to

$$\text{tr}\,(M_3'\{M_+, M_-\} + M_3\{M_+', M_-\} + M_3\{M_+, M_-'\}$$
$$= \text{tr}\,[b_3' b_1 b_2 + b_4' b_2 b_1 + b^3(b^1 b'^2 + b'^1 b^2) + b^4(b^2 b'^1 + b'^2 b^1)]$$
$$= n_g \frac{\varepsilon_2 \sin \Theta}{4 \cos \Theta}(\sin \Theta - 2\alpha), \tag{4.9}$$

with the same consequences as before.

*D. Case* $(3, +, -)_{AAA}$:
Now we have to compute the trace

$$\text{tr}\,(M_3'\{M_+', M_-'\} = \text{tr}\,[b'^3 b'^1 b'^2 + b'^4 b'^2 b'^1] = 0\,.$$

*E. Case* $(0, 0, 3)_{VVA}$:
Here the relevant trace is equal to

$$\text{tr}\,(M_0^2 M_3') = \text{tr}\,[(b^5)^2\, b'^3 + (b^6)^2\, b'^4 = n_g \varepsilon_2 \frac{\delta_3}{2} \sin \Theta (2\alpha - \sin \Theta)\,.$$

Due to the same factor as in case A, the compensation between leptons and quarks is the same. This is also true in the next case:

*F. Case* $(3, 3, 3)_{VVA}$:

$$\text{tr}\,(M_3^2 M_3') = \text{tr}\,[(b^3)^2\, b'^3 + (b^4)^2\, b'^4]$$
$$= n_g \varepsilon_2 \frac{\delta_3}{\cos \Theta} \left(\sin^3 \Theta - \frac{\sin \Theta}{2}\right)(2\alpha - \sin \Theta)\,.$$

*G. Case* $(3, 3, 3)_{AAA}$:
This final case is trivial:

$$\text{tr}\,(M_3')^3 = \text{tr}\,[(b'^3)^3 + (b'^4)^3] = 0\,.$$

Summing up the axial anomalies completely cancel in each generation, if and only if $\alpha_Q$ has the value (4.7).

## 5 Gauge invariance at third order: tree graphs

Third order tree graphs are not covered by the general inductive proof of gauge invariance in massless Yang-Mills theories [16]. They are part of the beginning of the induction and need an explicit verification of gauge invariance. The latter can easily be done in the massless Yang-Mills theory, by using the fact that all couplings are of Yang-Mills type, that means proportional to $f_{abc}$. But this is no longer true for the scalar couplings in massive Yang-Mills theories [1]. Therefore, it is not surprising that gauge invariance of third order tree diagrams fixes the last free parameters in the scalar coupling of the electroweak theory, namely the Higgs self-coupling (Higgs potential).

Let us first discuss a simple special case of the standard model, the $U(1)$ Higgs model, which contains all essential features of the scalar self-coupling. We consider one massive gauge field $W^\mu$ only, in interaction with one unphysical ($\Phi$) and one physical scalar field $\Phi_0$. Therefore, we let $W_1 = W$, $u_1 = u$, $\tilde{u}_1 = \tilde{u}$, $\Phi_1 = \Phi$, $m_W = m$ in (3.20) of [1] and omit the other fields. From (3.20.9, 10, 12) we then get

$$T_1 = i\,\frac{g}{2}\,\left\{ W^\nu(\Phi_0\,\partial_\nu\Phi - \Phi\,\partial_\nu\Phi_0) - mW_\nu W^\nu\Phi_0 \right.$$
$$\left. + \frac{m_H^2}{2m}\,\Phi_0\Phi^2 + m\tilde{u}u\Phi_0 + 2b\Phi_0^3 \right\}, \tag{5.1}$$

and from (4.5.10) of [1] or (3.2.9)

$$T_{1/1}^\mu|_{\text{an}} = -i\,\frac{g}{2}\,u\{\Phi\,\partial^\mu\Phi_0 - \Phi_0\,\partial^\mu\Phi\} \overset{\text{def}}{=} D_1 + D_2\,. \tag{5.2}$$

The calculation and compensation of the anomalies at second order can now be directly taken over from the appendix in [1]. Only the following two sectors 1) and 7) appear:

*1) Sector* $(\Phi_0, \Phi_0, 1, 1)$
Here we have found the two normalization terms (A.1, 2)

$$N_1 = \frac{i}{4}\,g^2 W_\nu W^\nu \Phi_0^2 \delta(x - y)\,, \tag{5.3}$$

$$N_2 = ig^2\left(\frac{m_H^2}{4m^2} - \frac{3b}{2m}\right)\Phi_0^2\Phi^2\delta(x - y)\,. \tag{5.4}$$

*7) Sector* $(1, 1, 1, 1)$

$$N_{10} = \frac{i}{4} g^2 W_\nu W^\nu \Phi^2 \delta(x - y) , \tag{5.5}$$

$$N_{11} = -ig^2 \frac{m_H^2}{16m^2} \Phi^4 \delta(x - y) . \tag{5.6}$$

These results $(5.3 - 6)$ are valid for the $U(1)$ Higgs model, too, because all additional couplings in the standard model do not contribute to these two sectors. Furthermore, the final normalization term (A.20)

$$N_{20} = ig^2 \lambda' \Phi_0^4 \delta(x - y) \tag{5.7}$$

becomes now important. Until now the coupling parameters $b$ (5.1) and $\lambda'$ (5.7) are arbitrary.

Gauge invariance (1.0) at third order can only be violated by local terms $\sim D\delta(x_1 - x_3, , x_2 - x_3)$ (where D denotes a differential operator). This can easily be seen by inserting the causal factorization of the time-ordered products and using gauge invariance at lower orders (cf. (4.3) in [1]). We only consider the tree diagrams

$$d_Q T_3|_{\text{tree}} = i \sum_{k=1, 2, 3} \partial_\nu^k T_{3/k}^\nu|_{\text{tree}} \tag{5.8}$$

and adopt the notations and terminology from the corresponding calculation at second order (sect. 4 of [1]).[4] There are no local terms in $T_3|_{\text{tree}}$ and $T_{3/k}|_{\text{tree}}$, because a term $\sim \delta(x_1 - x_3, x_2 - x_3) : B_1 B_2 B_3 B_4 B_5 :$ would violate renormalizability by power counting (terms with derivatives on $\delta(x_1 - x_3, x_2 - x_3)$ are even worse); especially there is no freedom of normalization (cf. (4.2) in [1]). Hence the only local terms in (5.8) are the anomalies which are the terms $\sim \delta(x_1 - x_3, x_2 - x_3)$ in $\sum_{k=1, 2, 3} \partial_\nu^k T_{3/k}^\nu|_{\text{tree}}(x_1, x_2, x_3)$. (Derivatives of the $\delta$-distribution do not appear as can be seen by power counting.) Up to permutation of the vertices there is only one possibility to generate an anomaly: due to causal factorization, e.g. $T_{3/1}^\mu(x_1, x_2, x_3) = T_{1/1}^\mu(x_1) T_2(x_2, x_3)$ for $x_1 \notin (\{x_2, x_3\} + \bar{V}^-)$, the normalization term $N_{(2)}(x_2, x_3) \sim \delta(x_2 - x_3)$ of $T_2|_{\text{tree}}(x_2, x_3)$ is contained in $T_{3/1}^\mu(x_1, x_2, x_3)$. Hence, if $\partial^\mu \Phi_0$ in $D_1$, or $\partial^\mu \Phi$ in $D_2$ (5.2), is contracted with a field operator in $N_{(2)}(x_2, x_3)$ we obtain a term

$$T_{3/1}|_{\text{tree}}^\mu(x_1, x_2, x_3)$$
$$= \sim \partial^\mu D_F(x_1 - x_2) \delta(x_2 - x_3) : B_1(x_1) B_2(x_1) B_3(x_2) B_4(x_3) B_5(x_3) : + \dots .$$

Taking now the divergence with respect to the $Q$-vertex $x_1$, an anomaly appears due to $\partial_\mu \partial^\mu D_F(x_1 - x_2) = -m^2 D_F(x_1 - x_2) + \delta(x_1 - x_2)$. Considering the external field operators $u\Phi\Phi_0^3$ there come anomalies from the combinations of $D_1$ with $N_{20}$ and of $D_2$ with $N_2$. Gauge invariance requires the cancellation of these anomalies

$$(D_1, N_{20})_{\text{loc}} + (D_2, N_2)_{\text{loc}} = 0 ,$$

---

[4] Note that e.g. a term $\sim \delta(x_1 - x_2) \partial_\mu \partial_\nu D_F(x_2 - x_3)$ ($\mu$ and $\nu$ not contracted) is non-local.

which gives the condition

$$-i\,\frac{g}{2}\,4ig^2\lambda' + i\,\frac{g}{2}\,2ig^2\left(\frac{m_H^2}{4m^2} - \frac{3b}{2m}\right) = 0\,. \tag{5.9}$$

The factors 4 and 2 are due to the number of possible contractions between the two members $D$ and $N$. Because $N_{20}$ and $N_2$ depend on the unknown parameters $\lambda'$ and $b$, this is a first equation to determine these parameters.

The anomalies with field operators $u\Phi\Phi_0 W_\nu W^\nu$ come from the terms

$$(D_1, N_1)_{\text{loc}} + (D_2, N_{10})_{\text{loc}} = 0\,.$$

This gives the condition

$$-i\,\frac{g}{2}\,2i\,\frac{g^2}{4} + i\,\frac{g}{2}\,2\,\frac{i}{4}\,g^2 = 0\,,$$

which is automatically satisfied. Finally, in the sector $u\Phi_0\Phi^3$ gauge invariance requires

$$(D_1, N_2)_{\text{loc}} + (D_2, N_{11})_{\text{loc}} = 0$$

and this leads to the constraint

$$-i\,\frac{g}{2}\,2ig^2\left(\frac{m_H^2}{4m^2} - \frac{3b}{2m}\right) + i\,\frac{g}{2}\,4(-ig^2)\,\frac{m_H^2}{16m^2} = 0\,. \tag{5.10}$$

From (5.9) and (5.10) we find the values of the two coupling parameters

$$b = \frac{m_H^2}{4m}\,, \qquad \lambda' = -\frac{m_H^2}{16m^2}\,. \tag{5.11}$$

The same argument applies to the whole electroweak theory: In the sector $u_1\Phi_1\Phi_0^3$ we get the equation (5.9) (with $m = m_W$), and in the sector $u_1\Phi_0\Phi_1^3$ we find the condition (5.10) again. It is easy to see that there are no additional terms contributing in the bigger theory. The resulting values

$$b = \frac{m_H^2}{4m_W}\,, \qquad \lambda' = -\frac{m_H^2}{16m_W^2} \tag{5.12}$$

agree with those obtained from the Higgs potential in the standard theory as shown below. We have verified that, with these parameters, all other anomalies from third order tree graphs cancel out, without giving further information.

For comparison with the standard model we collect all self-couplings of scalar fields. The first order terms are contained in $(3.20.10-12)$ of [1]:

$$V_1(\Phi) = ig\,\frac{m_H^2}{4m_W}\,\Phi_0(\Phi_0^2 + \Phi_1^2 + \Phi_2^2 + \Phi_3^2)\,. \tag{5.13}$$

The second order terms are given by the normalization terms in the appendix of [1]

$$V_2(\Phi) = ig^2\,\frac{m_H^2}{16m_W^2}\,[-(\Phi_0^2 + \Phi_1^2 + \Phi_2^2 + \Phi_3^2)^2]\,. \tag{5.14}$$

Remembering that the second order must be multiplied by 1/2, we obtain the following total scalar potential

$$V(\Phi) = V_1 + \frac{1}{2}\, V_2 = -ig^2\, \frac{m_H^2}{32 m_W^2}$$

$$\times \left[ (\Phi_0^2 + \Phi_1^2 + \Phi_2^2 + \Phi_3^2)^2 - 8\, \frac{m_W}{g}\, \Phi_0(\Phi_0^2 + \Phi_1^2 + \Phi_2^2 + \Phi_3^2) \right]. \quad (5.15)$$

To compare this with the Goldstone-Higgs potential of the standard theory we set $\Phi_1 = \Phi_2 = \Phi_3 = 0$ and add a mass term $\frac{1}{2}\, m_H^2 \Phi_0^2$. Omitting the factor $-i$, the potential is then equal to

$$V(\Phi_0) = \lambda(\Phi_0^4 - 4a\Phi_0^3 + 4a^2\Phi_0^2)\,, \quad (5.16)$$

where

$$\lambda = \frac{1}{2} \left( \frac{gm_H}{4m_W} \right)^2, \qquad a = \frac{2}{g}\, m_W\,. \quad (5.17)$$

This is the shifted double-well potential discussed in the introduction (1.1). Our point is that this structure is not obtained from a clever choice of a Lagrangean and subsequent symmetry breaking, but comes out as the necessary consequence of gauge invariance in the massive situation.

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