
Algebraic Structures and Complete Vector Spaces

Definition 1 Let G be a set with a binary internal operation $\circ : G \times G \rightarrow G$. $(G, \circ)$ is called a group if the following properties hold:

- (M) $a, b \in G \Rightarrow \circ(a, b) =: a \circ b \in G$
- (A) $a \circ (b \circ c) = (a \circ b) \circ c \quad \forall a, b, c \in G$ (Associativity)
- (N) $\exists_1 n \in G, n \circ a = a \circ n = a \quad \forall a \in G$ (Existence of the unique neutral element)
- (I) $a \in G \Rightarrow \exists_1 a^{-1} \in G, a^{-1} \circ a = a \circ a^{-1} = n$ (Inverse element)

If property (M) holds for G , then $(G, \circ)$ is called a *magma*. Examples: $(\mathbb{Z}, -)$, $(\mathbb{N}, a \circ b = a^b)$.

If properties (M) and (A) hold in G , then $(G, \circ)$ is a *semigroup*. Example: $(\mathbb{N}, +)$.

If $(G, \circ)$ is a semigroup with a neutral element according to (N), then $(G, \circ)$ is called a *monoid*. E.g.: $(\mathbb{N}, \cdot)$.

A group $(G, \circ)$ is called *abelian* if $a \circ b = b \circ a$ holds for all $a, b \in G$.

Definition 2 Let K be a set with two binary operations $\oplus : K \times K \rightarrow K$ and $\odot : K \times K \rightarrow K$. $(K, \oplus, \odot)$ is called a field if the following conditions hold:

- (K1) $(K, \oplus)$ is an abelian group with neutral element 0.
- (K2) $(K \setminus \{0\}, \odot)$ is an abelian group with neutral element 1.
- (K3) $\forall a, b, c \in K: a \odot (b \oplus c) = a \odot b \oplus a \odot c, (a \oplus b) \odot c = a \odot c \oplus b \odot c$ (distributive law).

Notation rule: $\odot$ is performed before $\oplus$ ("multiplication before addition").

Examples: The rational numbers $(\mathbb{Q}, +, \cdot)$, the real numbers $(\mathbb{R}, +, \cdot)$, the complex numbers $(\mathbb{C}, +, \cdot)$, meromorphic functions on a domain in the complex plane with the usual addition and multiplication.

Definition 3 A vector space V_K over a field $(K, \oplus, \odot)$ is an additive abelian group $(V_K, +)$, on which in addition a multiplication $\star : K \times V_K \rightarrow V_K$ with a scalar from K is defined.

For all vectors $u, v \in V_K$ and scalars $\alpha, \beta \in K$ the following defining axioms (V1-4) hold:

- (V1) $\alpha \star (\beta \star v) = (\alpha \odot \beta) \star v$
- (V2) $\alpha \star (u + v) = \alpha \star u + \alpha \star v$
- (V3) $(\alpha \oplus \beta) \star v = \alpha \star v + \beta \star v$
- (V4) $1 \star v = v$, where 1 denotes the unity element of the field K .

In quantum mechanics, real and complex vector spaces ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) play an important role.

Examples: $\mathbb{R}_{\mathbb{R}}^n, \mathbb{C}_{\mathbb{C}}^n, \mathbb{C}_{\mathbb{R}}^n, C^k(\mathbb{R}) = \{f : \mathbb{R} \rightarrow \mathbb{R} \text{ or } \mathbb{C} \mid f \text{ k-times continuously differentiable on } \mathbb{R}\}$.

Remark: If a vector space is equipped with an additional vector-valued second operation (vector multiplication) between two vectors, one often speaks of an *algebra*. Sets with operation structures are generally called *algebraic structures*.

Examples: 3 dimensional Euclidean vector space with vector addition and vector product ("cross product"), quaternion algebra $\mathbb{H}_{\mathbb{R}}$, octonions $\mathbb{O}_{\mathbb{R}}$, complex $n \times n$ matrices $Mat(n, \mathbb{C})$ with corresponding operations.

Remark: A scalar is thus an element of the base field of a vector space V_K , while a vector is an element of V_K . The term *scalar* is also used in physics in another sense for a quantity independent of the inertial frame of an observer (e.g. the rest mass of a particle).

Definition 4 n vectors $\{v_1, \dots, v_n\} \subset V_K$ are called linearly independent if for $\{\lambda_1, \dots, \lambda_n\} \subset K$

$$\lambda_1 \star v_1 + \dots + \lambda_n \star v_n = \mathbf{0} \quad \Leftrightarrow \quad \lambda_1 = \dots = \lambda_n = 0$$

holds. Here the zero vector $\mathbf{0}$ is the neutral element in $(V_K, +)$.

Definition 5 n linearly independent vectors $\{v_1, \dots, v_n\} \subset V_K$ form a basis of V_K , if every vector $v \in V_K$ can be written as the following linear combination:

$$v = \mu_1 \star v_1 + \dots + \mu_n \star v_n, \quad \{\mu_1, \dots, \mu_n\} \subset K.$$

$n = \dim(V_K)$ is the dimension of V_K . If the vector space consists only of the element $\mathbf{0}$, then $\dim(\mathbf{0}) = 0$.

Examples: $\dim(\mathbb{R}^n) = n$, $\dim(\mathbb{C}^n) = n$, $\dim(\mathbb{C}^n_{\mathbb{R}}) = 2n$, $\dim(C^0([0, 1])) = \infty$ in some sense.

Definition 6 A norm is a mapping $\|\cdot\| : V_K \rightarrow \mathbb{R}_0^+$ from a vector space over the field $\mathbb{K}$ of real or complex numbers into the non-negative real numbers $\mathbb{R}_0^+$, which satisfies for all vectors $x, y \in V_K$ and all scalars $\lambda \in \mathbb{K}$ the following axioms:

$$(N1) \quad \|x\| = 0 \Leftrightarrow x = 0 \quad (\text{Definiteness})$$

$$(N2) \quad \|\lambda x\| = |\lambda| \cdot \|x\|, \quad |\lambda| = \sqrt{\bar{\lambda} \cdot \lambda} \geq 0 \quad (\text{Homogeneity})$$

$$(N3) \quad \|x + y\| \leq \|x\| + \|y\| \quad (\Delta\text{-inequality or subadditivity})$$

As above, the symbols $+$ and $\cdot$ will be used for both scalar and vector addition and multiplication. $\bar{\lambda} = \lambda^* = \lambda_1 - i \cdot \lambda_2$ denotes the complex conjugate of $\lambda = \lambda_1 + i\lambda_2$, $\lambda_{1,2} \in \mathbb{R}$, $i^2 = -1$.

A vector space with a norm is called a *normed vector space*.

Definition 7 A Banach space $\mathcal{B}$ is a complete normed vector space, i.e. in such a space every Cauchy sequence $\{a_n\}_{n \in \mathbb{N}} \subset \mathcal{B}$ (with the property that $\forall \epsilon > 0 \exists N(\epsilon)$ such that $\|a_n - a_m\| < \epsilon$ for all $n, m \geq N(\epsilon)$) has a (unique) limit $a \in \mathcal{B}$; that is, for this a , $\forall \epsilon > 0 \exists N(\epsilon)$ such that $\|a_m - a\| < \epsilon$ for all $m \geq N(\epsilon)$. A Cauchy sequences is also called a *fundamental sequence* or *convergent in itself*.

The first systematic investigation of Banach spaces is found in the dissertation of Stefan Banach (1922).

Definition 8 Let V_K be a vector space over the real or complex numbers. An inner product or scalar product is a positive definite Hermitian sesquilinear form (“one-and-a-half linear”), i.e. a mapping $\langle \cdot, \cdot \rangle : V_K \times V_K \rightarrow \mathbb{K}$, which satisfies the following conditions for all $x, y, z \in V_K$ and $\lambda \in \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$:

$$(I1) \quad \langle x, x \rangle \geq 0 \quad (\text{positive definiteness})$$

$$(I2) \quad \langle x, x \rangle = 0 \Leftrightarrow x = 0 \quad (\text{definiteness})$$

$$(I3) \quad \langle x, y \rangle = \overline{\langle y, x \rangle} \quad (\text{symmetry, Hermiticity})$$

$$(I4) \quad \langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle \quad (\text{linearity})$$

From these axioms it immediately follows that the first argument is antilinear: $\langle \lambda x, y \rangle = \bar{\lambda} \langle x, y \rangle$.

Mathematicians often require linearity in the left rather than in the right argument in (I4).

Definition 9 A vector space with inner product $(V_K, \langle \cdot, \cdot \rangle)$ is called a *pre-Hilbert space*. In the real case one also speaks of a Euclidean space $V_{\mathbb{R}}$, in the complex case of a unitary space $V_{\mathbb{C}}$. These spaces are also normed vector spaces with the induced norm $\|x\| = \sqrt{\langle x, x \rangle} \geq 0$.

Definition 10 A Hilbert space is a Euclidean or unitary vector space which is complete with respect to the norm induced by the scalar product. Often only the complex (unitary) case is considered in physics. An incomplete pre-Hilbert space can always be uniquely (up to isomorphism) extended to a Hilbert space by adding missing limits in the form of equivalence classes of corresponding Cauchy sequences as abstract elements to the pre-Hilbert space.