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5 November 2001

PHYSICS LETTERS A

Physics Letters A 290 (2001) 106–108

www.elsevier.com/locate/pla

Comment

Comment on “Another form of the Klein–Gordon equation”

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Received 20 August 2001; accepted 24 September 2001
Communicated by P.R. Holland

Abstract

It is shown that the alternative Klein–Gordon equation with positive definite probability density proposed in a Letter by M.D. Kostin does not meet the requirements of relativistic (quantum) field theory and therefore does not allow for a meaningful physical interpretation. © 2001 Elsevier Science B.V. All rights reserved.

PACS: 11.10.-z
Keywords: Klein–Gordon equation; Relativistic field theory

The alternative formulation of the Klein–Gordon equation [1,2] proposed by Kostin [3] reads

$$i\hbar\frac{\partial\phi}{\partial t} = +mc^2\phi + c\big(\hat{\vec{p}}\vec{\psi}\big), \tag{1}$$

$$i\hbar\frac{\partial\vec{\psi}}{\partial t} = -mc^2\vec{\psi} + c\hat{\vec{p}}\phi, \tag{2}$$

where $\phi(\vec{r},t)$ and $\vec{\psi}(\vec{r},t)$ are “scalar” and “vector” probability amplitudes, respectively, and $\hat{\vec{p}} = -i\hbar\vec{\nabla}$. Defining the probability density

$$P = \phi^*\phi + \big(\vec{\psi}^*\vec{\psi}\big) \tag{3}$$

and the probability current density

$$\vec{S} = c\big(\phi^*\vec{\psi} + \phi\vec{\psi}^*\big), \tag{4}$$

one readily derives the probability conservation equation

$$\frac{\partial P}{\partial t} + \vec{\nabla}\vec{S} = 0. \tag{5}$$

It is a nice feature of the probability density $P(\vec{r},t)$ to be positive definite, although it is clear that the non-existence of a positive definite probability density for the Klein–Gordon equation is no more a problem in quantum field theory.

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PII: S0375-9601(01)00631-4

Multiplying (1) with $i\hbar(\partial/\partial t) + mc^2$ and (2) by $c\hat{\vec{p}}$ and combining the results, one obtains

$$\hbar^2 \frac{\partial^2}{\partial t^2}\phi - c^2\hbar^2\vec{\nabla}^2\phi + m^2c^4\phi = 0, \tag{6}$$

i.e., ϕ satisfies the Klein–Gordon equation, but in a similar way one immediately sees that the components of $\vec{\psi}$ fulfill the (non-covariant) equation

$$\hbar^2 \frac{\partial^2}{\partial t^2}\vec{\psi} - c^2\hbar^2\vec{\nabla}\big(\vec{\nabla}\vec{\psi}\big) + m^2c^4\vec{\psi} = 0. \tag{7}$$

Although the problematic nature of Eqs. (1)–(5) can be uncovered easily, their tempting form sometimes leads to confusion and the equations have even found their way into literature [4]. Furthermore, when the scalar particle described by $(\phi, \vec{\psi})$ is coupled to an electromagnetic potential, different results are obtained as in the case of the Klein–Gordon equation. One must therefore ask if the proposed equations should be treated on an equal footing with the usual Klein–Gordon equation.

We give simple arguments in the following which show that the alternative form of the Klein–Gordon equation is hard to interpret in a meaningful way. Obviously, (1) and (2) can be cast in a Dirac-like form

$$i\hbar\frac{\partial}{\partial t}\Psi = mc^2\beta\Psi + c\big(\vec{\alpha}\,\vec{p}\big)\Psi, \tag{8}$$

with appropriate matrices β and $\vec{\alpha}$ and the four-component wave function

$$\Psi = \begin{pmatrix} \phi \\ \vec{\psi} \end{pmatrix}, \tag{9}$$

or, using a more compact notation, in the following where $\hbar = c = 1$:

$$\big\{i\gamma^\mu\partial_\mu - m\big\}\Psi(x) = \big\{\gamma^\mu\hat{P}_\mu - m\big\}\Psi(x) = 0. \tag{10}$$

Then it is easy to show by straightforward calculation that matrices $S(\Lambda)$ which relate the wave functions in different coordinates x, x',

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu, \qquad x^\nu = \big(ct, \vec{r}\big), \qquad \Lambda^\mu{}_\rho\gamma^\rho = S^{-1}(\Lambda)\gamma^\mu S(\Lambda), \tag{11}$$

according to

$$\Psi'(x') = S(\Lambda)\Psi(x) = S(\Lambda)\Psi\big(\Lambda^{-1}x'\big), \tag{12}$$

$$\big\{\gamma^\mu\hat{P}_\mu - m\big\}\Psi(x) = \big\{\gamma^\mu\hat{P}'_\mu - m\big\}\Psi'(x') = 0, \tag{13}$$

exist trivially for rotations, but not for general Lorentz transformations [5].

A severe problem arises when one considers the propagators for the proposed theory. The Dirac equation can be written in an explicit form as follows:

$$\begin{pmatrix} \hat{p}_0 - m & 0 & \hat{p}_3 & \hat{p}_1 - i\hat{p}_2 \\ 0 & \hat{p}_0 - m & \hat{p}_1 + i\hat{p}_2 & -\hat{p}_3 \\ -\hat{p}_3 & -\hat{p}_1 + i\hat{p}_2 & -\hat{p}_0 - m & 0 \\ -\hat{p}_1 - i\hat{p}_2 & \hat{p}_3 & 0 & -\hat{p}_0 - m \end{pmatrix} \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix}_D = 0, \tag{14}$$

and by formal inversion of the matrix in (14) the retarded (advanced) propagator can be constructed in momentum space

$$\tilde{S}_{R,A}(p) \sim \frac{\gamma^\mu p_\mu + m}{p^2 - m^2 \pm ip_0 0}, \tag{15}$$

which has causal support in real space

$$\mathrm{supp}\big(S_{R,A}(x)\big) \subseteq V^{\pm}, \tag{16}$$

$$V^{+} = \big\{x \in \mathbf{R}^4 \mid x^2 \geqslant 0,\ x^0 \geqslant 0\big\}, \qquad V^{-} = \big\{x \in \mathbf{R}^4 \mid x^2 \geqslant 0,\ x^0 \leqslant 0\big\}, \tag{17}$$

a fact which expresses, roughly speaking, the causal structure of the theory [6]. The support property (16) of the tempered distributions $S_{R,A} \in \mathcal{S}'(\mathbf{R}^4)$ means that the product $\langle S_{R,A} \mid f\rangle$ vanishes for all rapidly decreasing test functions in Schwartz space $f \in \mathcal{S}(\mathbf{R}^4)$ which have their support outside the forward (backward) light cone. But in the present case, inversion of the differential operator

$$\begin{pmatrix} \hat{p}_0 - m & -\hat{p}_1 & -\hat{p}_2 & -\hat{p}_3 \\ \hat{p}_1 & -\hat{p}_0 - m & 0 & 0 \\ \hat{p}_2 & 0 & -\hat{p}_0 - m & 0 \\ \hat{p}_3 & 0 & 0 & -\hat{p}_0 - m \end{pmatrix} \tag{18}$$

leads to a result

$$\sim \frac{1}{p^2 - m^2} \begin{pmatrix} p_0 - m & -p_1 & -p_2 & -p_3 \\ p_1 & \frac{-p_0^2+p_2^2+p_3^2+m^2}{p_0+m} & -\frac{p_1p_2}{p_0+m} & -\frac{p_1p_3}{p_0+m} \\ p_2 & -\frac{p_1p_2}{p_0+m} & \frac{-p_0^2+p_1^2+p_3^2+m^2}{p_0+m} & -\frac{p_2p_3}{p_0+m} \\ p_3 & -\frac{p_1p_3}{p_0+m} & \frac{-p_2p_3}{p_0+m} & \frac{-p_0^2+p_1^2+p_2^2+m^2}{p_0+m} \end{pmatrix}, \tag{19}$$

which is in conflict with the requirements of the local structure of quantum field theory due to the non-local operator $\sim(\hat{p}_0 + m)^{-1}$ in the propagator. The description of a scalar particle in the Duffin–Kemmer–Petiau formalism [7–9] by a five-component wave function is equivalent (at least on the classical level) to the usual Klein–Gordon equation and causes no problems of that kind [10].

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