Lecture Notes

# Mathematical Methods of Particle Physics


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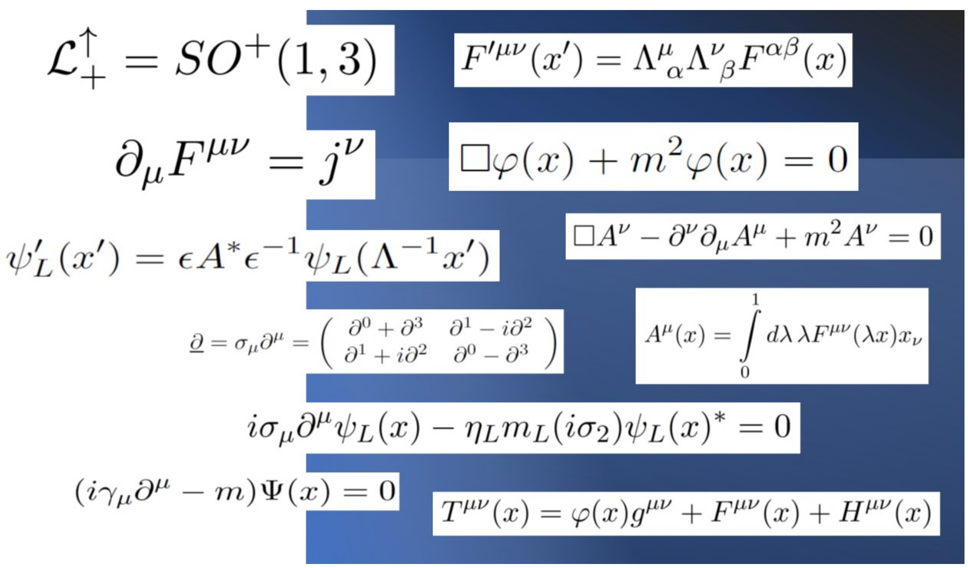


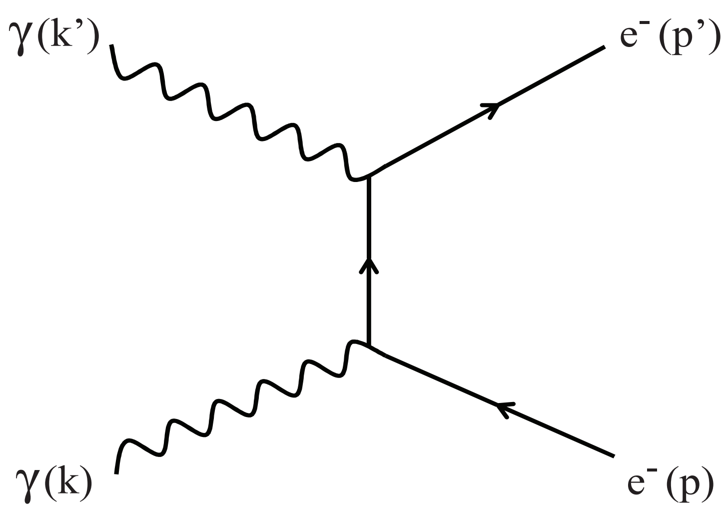

# Algebraic Structures and Complete Vector Spaces

**Definition 1** *Let $G$ be a set with a binary internal operation $\circ : G \times G \to G$. $(G, \circ)$ is called a group if the following properties hold:*

*(M)* $a, b \in G \Rightarrow \circ(a, b) =: a \circ b \in G$

*(A)* $a \circ (b \circ c) = (a \circ b) \circ c \quad \forall a, b, c \in G$ *(Associativity)*

*(N)* $\exists_1\, n \in G, n \circ a = a \circ n = a \quad \forall a \in G$ *(Existence of the unique neutral element)*

*(I)* $a \in G \Rightarrow \exists_1\, a^{-1} \in G,\ a^{-1} \circ a = a \circ a^{-1} = n$ *(Inverse element)*

If property $(M)$ holds for $G$, then $(G, \circ)$ is called a *magma*. Examples: $(\mathbb{Z}, -)$, $(\mathbb{N}, a \circ b = a^b)$.
If properties $(M)$ and $(A)$ hold in $G$, then $(G, \circ)$ is a *semigroup*. Example: $(\mathbb{N}, +)$.
If $(G, \circ)$ is a semigroup with a neutral element according to $(N)$, then $(G, \circ)$ is called a *monoid*. E.g.: $(\mathbb{N}, \cdot)$.

A group $(G, \circ)$ is called *abelian* if $a \circ b = b \circ a$ holds for all $a, b \in G$.

**Definition 2** *Let $K$ be a set with two binary operations $\oplus : K \times K \to K$ and $\odot : K \times K \to K$.*
*$(K, \oplus, \odot)$ is called a field if the following conditions hold:*

*(K1)* $(K, \oplus)$ *is an abelian group with neutral element* $0$.

*(K2)* $(K \backslash 0, \odot)$ *is an abelian group with neutral element* $1$.

*(K3)* $\forall a, b, c \in K$: $a \odot (b \oplus c) = a \odot b \oplus a \odot c$, $(a \oplus b) \odot c = a \odot c \oplus b \odot c$ *(distributive law).*

Notation rule: $\odot$ is performed before $\oplus$ (”multiplication before addition”).

**Examples**: The rational numbers $(\mathbb{Q}, +, \cdot)$, the real numbers $(\mathbb{R}, +, \cdot)$, the complex numbers $(\mathbb{C}, +, \cdot)$, meromorphic functions on a domain in the complex plane with the usual addition and multiplication.

**Definition 3** *A vector space $V_K$ over a field $(K, \oplus, \odot)$ is an additive abelian group $(V_K, +)$, on which in addition a multiplication $\star :\ K \times V_K \to V_K$ with a scalar from $K$ is defined.*
*For all vectors $u,\ v \in V_K$ and scalars $\alpha,\ \beta \in K$ the following defining axioms (V1-4) hold:*

*(V1)* $\alpha \star (\beta \star v) = (\alpha \odot \beta) \star v$

*(V2)* $\alpha \star (u + v) = \alpha \star u + \alpha \star v$

*(V3)* $(\alpha \oplus \beta) \star v = \alpha \star v + \beta \star v$

*(V4)* $1 \star v = v$, *where* $1$ *denotes the unity element of the field* $K$.

In quantum mechanics, real and complex vector spaces ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) play an important role.

**Examples**: $\mathbb{R}^n_{\mathbb{R}}$, $\mathbb{C}^n_{\mathbb{C}}$, $\mathbb{C}^n_{\mathbb{R}}$, $C^k(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \text{ or } \mathbb{C} \,|\, f\ k\text{-times continuously differentiable on } \mathbb{R}\}$.

**Remark**: If a vector space is equipped with an additional vector-valued second operation (vector multiplication) between two vectors, one often speaks of an *algebra*. Sets with operation structures are generally called *algebraic structures*.

**Examples**: 3 dimensional Euclidean vector space with vector addition and vector product (”cross product”), quaternion algebra $\mathbb{H}_{\mathbb{R}}$, octonions $\mathbb{O}_{\mathbb{R}}$, complex $n \times n$ matrices $Mat(n, \mathbb{C})$ with corresponding operations.

**Remark**: A scalar is thus an element of the base field of a vector space $V_K$, while a vector is an element of $V_K$. The term *scalar* is also used in physics in another sense for a quantity independent of the inertial frame of an observer (e.g. the rest mass of a particle).

**Definition 4** *$n$ vectors $\{v_1, \ldots, v_n\} \subset V_K$ are called* linearly independent *if for $\{\lambda_1, \ldots, \lambda_n\} \subset K$*

$$\lambda_1 \star v_1 + \ldots + \lambda_n \star v_n = \mathbf{0} \quad \Leftrightarrow \quad \lambda_1 = \ldots = \lambda_n = 0$$

*holds. Here the zero vector $\mathbf{0}$ is the neutral element in $(V_K, +)$.*

**Definition 5** *$n$ linearly independent vectors $\{v_1, \ldots, v_n\} \subset V_K$ form a basis of $V_K$, if every vector $v \in V_K$ can be written as the following linear combination:*

$$v = \mu_1 \star v_1 + \ldots + \mu_n \star v_n\,, \quad \{\mu_1, \ldots, \mu_n\} \subset K\,.$$

$n = \dim(V_K)$ is the dimension of $V_K$. If the vector space consists only of the element $\mathbf{0}$, then $\dim(\mathbf{0}) = 0$.

**Examples**: $\dim(\mathbb{R}^n_{\mathbb{R}}) = n$, $\dim(\mathbb{C}^n_{\mathbb{C}}) = n$, $\dim(\mathbb{C}^n_{\mathbb{R}}) = 2n$, $\dim(C^0([0,1])) = \infty$ in some sense.

**Definition 6** *A norm is a mapping $||\cdot|| : V_{\mathbb{K}} \to \mathbb{R}^+_0$ from a vector space over the field $\mathbb{K}$ of real or complex numbers into the non-negative real numbers $\mathbb{R}^+_0$, which satisfies for all vectors $x,\, y \in V_{\mathbb{K}}$ and all scalars $\lambda \in \mathbb{K}$ the following axioms:*

*(N1)* $||x|| = 0 \Leftrightarrow x = 0$ *(Definiteness)*

*(N2)* $||\lambda x|| = |\lambda| \cdot ||x||\,, \quad |\lambda| = \sqrt{\overline{\lambda} \cdot \lambda} \geq 0$ *(Homogeneity)*

*(N3)* $||x + y|| \leq ||x|| + ||y||$ *($\Delta$-inequality or subadditivity)*

As above, the symbols $+$ and $\cdot$ will be used for both scalar and vector addition and multiplication. $\overline{\lambda} = \lambda^* = \lambda_1 - i \cdot \lambda_2$ denotes the complex conjugate of $\lambda = \lambda_1 + i\lambda_2$, $\lambda_{1,2} \in \mathbb{R}$, $i^2 = -1$.
A vector space with a norm is called a *normed vector space*.

**Definition 7** *A Banach space $\mathcal{B}$ is a complete normed vector space, i.e. in such a space every Cauchy sequence $\{a_n\}_{n \in \mathbb{N}} \subset \mathcal{B}$ (with the property that $\forall\, \epsilon > 0\; \exists\; N(\epsilon)$ such that $||a_n - a_m|| < \epsilon$ for all $n, m \geq N(\epsilon)$) has a (unique) limit $a \in \mathcal{B}$; that is, for this $a$, $\forall\, \epsilon > 0\; \exists\; N(\epsilon)$ such that $||a_m - a|| < \epsilon$ for all $m \geq N(\epsilon)$. A Cauchy sequences is also called a fundamental sequence or convergent in itself.*

The first systematic investigation of Banach spaces is found in the dissertation of Stefan Banach (1922).

**Definition 8** *Let $V_{\mathbb{K}}$ be a vector space over the real or complex numbers. An inner product or scalar product is a positive definite Hermitian sesquilinear form ("one-and-a-half linear"), i.e. a mapping $\langle \cdot, \cdot \rangle : V_{\mathbb{K}} \times V_{\mathbb{K}} \to \mathbb{K}$, which satisfies the following conditions for all $x,\, y,\, z \in V_{\mathbb{K}}$ and $\lambda \in \mathbb{K} = \mathbb{R}$ or $\mathbb{C}$:*

*(I1)* $\langle x, x \rangle \geq 0$ *(positive definiteness)*

*(I2)* $\langle x, x \rangle = 0 \Leftrightarrow x = 0$ *(definiteness)*

*(I3)* $\langle x, y \rangle = \overline{\langle y, x \rangle}$ *(symmetry, Hermiticity)*

*(I4)* $\langle x, \lambda y + z \rangle = \lambda \langle x, y \rangle + \langle x, z \rangle$ *(linearity)*

From these axioms it immediately follows that the first argument is antilinear: $\langle \lambda x, y \rangle = \bar{\lambda} \langle x, y \rangle$.
Mathematicians often require linearity in the left rather than in the right argument in *(I4)*.

**Definition 9** *A vector space with inner product $(V_{\mathbb{K}}, \langle \cdot, \cdot \rangle)$ is called a pre-Hilbert space. In the real case one also speaks of a* Euclidean space $V_{\mathbb{R}}$, *in the complex case of a* unitary space $V_{\mathbb{C}}$. *These spaces are also normed vector spaces with the induced norm $||x|| = \sqrt{\langle x, x \rangle} \geq 0$.*

**Definition 10** *A Hilbert space is a Euclidean or unitary vector space which is* complete *with respect to the norm induced by the scalar product. Often only the complex (unitary) case is considered in physics. An incomplete pre-Hilbert space can always be uniquely (up to isomorphism) extended to a Hilbert space by adding missing limits in the form of equivalence classes of corresponding Cauchy sequences as abstract elements to the pre-Hilbert space.*